HOW DO CURRENT TERM STRUCTURE MODELS BEHAVE BEYOND THE LAST LIQUID POINT?

A comparison of the DNS and Smith-Wilson methods

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January 2015

Abstract

This paper compares the popular Dynamic Nelson-Siegel (DNS) model with the Smith-Wilson (SW) method for the extrapolation of yield curves within the scope of the new regulation for pension funds and insurance companies, Solvency II. I have focused particularly on the behavior of the models after the last liquid point (LLP) of observable data. My main research shows that a longer LLP is beneficial at extrapolating the yield curve as well as using a convergence period that relies on the available data. I also found that the DNS model is more market consistent whereas the SW method performs better fitting the available data and disregards the information they provide at the long-end of the curve.

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1. Introduction

Without a doubt the term structure of interest rates has been the subject of much debate, and interest rates themselves play a huge role for many economic agents – they are a crucial macroeconomic variable, and they can be used as a monetary policy instrument and a tool for determining the time value of money. Given the recent events in financial markets many of the behaviors observed a decade ago have changed. In Europe this was not only caused by the 2008 financial crisis but also by the 2011 sovereign debt crisis, where one consequence was the appearance of yield curves that were far from normal and the widening of credit spreads between Euro area members. So from this point of view there is some interest in having a better understanding of how the term structure behaves.

Moreover, this discussion has become of particular importance with the introduction of new regulations for insurers' and pension funds' capital requirements in the scope of Solvency II. In the current Solvency II directive, article 75 states that assets and liabilities have to be valued with a market-consistent approach. At the same time, the insurers and pension funds liable to this new regulation have liabilities that mature far beyond anything traded in the market. So a issue appears: how can a reliable discount curve be computed when liquid data is bound to lower maturities than those of the liabilities to be discounted?

1.1 Research Question

The aforementioned issue brings me to my problem statement: "How do current term structure models behave beyond the last liquid point?" with the last liquid point (henceforth LLP) being defined as the maturity at which observed data is assumed to run out. Available market data tends to be less reliable at this end of the curve due to lack of liquidity (thin market problem) or even inexistent instruments with those maturities. The latest guidelines for implementation of Solvency II suggest the LLP should be 20 years (EIOPA, 2014). Hence the goal of this thesis is to compare the literature that models the yield curve and extrapolates it, in particular the dynamic framework of the popular Nelson-Siegel model, to the method proposed in the Solvency II guidelines (the Smith-Wilson method and the Ultimate Forward Rate).

To better understand the behavior of these models I will guide my research by answering several sub-questions. Firstly, the Netherlands have applied regulations similar to those in Solvency II and some studies have shown that possibly the chosen LLP of 20 years might be too early and too abrupt (Rebel, 2012). There are two issues here: one is the uncertainty on how long convergence should take between the market data and a pre-determined interest rate level (in this case, the Ultimate Forward Rate). Hence one of the concerns I would like to address is whether stability is achieved at a maturity, T2, similar for both approaches. The second issue raised is whether the 20-year LLP is in fact the correct one – there is evidence that market data can be liquid up to the 30-year point, so this is something I will also be testing.

When it comes to the models being compared there are also some concerns to be answered. Eventually both models will converge to a constant level, but it would be interesting to see how much the dynamic Nelson-Siegel derived curves would deviate on the long-end compared to the Smith-Wilson curves. Furthermore when comparing the models it is important to keep their purpose in mind – I am looking for a model that can produce a realistic discount curve to use in the scope of Solvency II. So I will also be looking into which model provides the better approach for this purpose.

Finally, I will also discuss the current level set for the Ultimate Forward Rate (henceforth UFR) of 4.2%. The UFR is based off the idea of mean reversion but one could argue on the feasibility of such a high level given the yields observable in the market in the recent past as well as the inflation and economic growth expectations.

Thus I will have 4 main hypotheses guiding my research:

1. Is the maturity at which stability is achieved (T2) the same for the models being tested?
2. How does the long-end of the curve compare in the NS model versus the SW-UFR approach?
3. Which model do we expect to provide a superior approach for the purpose of Solvency II?
4. Is the pre-determined UFR level of 4.2% realistic?

1.2 Literature

The Nelson-Siegel class of models in which I am focusing this paper builds upon the traditional Expectations Theory of the term structure, which states that the yield of an *n*-period bond will be an average of all "expected" yields over the next *n* periods. This theory is a little simplistic and many have added to this proposition such as Cox, Ingersoll, and Ross (1981) with the Preferred Habitat Theory, and Hicks (1946) with the Liquidity Preference Theory.

The Nelson-Siegel model (Nelson & Siegel, 1987) aims to describe the yield curve taking into account 3 factors – level, slope and curvature. Plenty of literature has transformed the original model by incorporating some alterations like a no arbitrage assumption, the introduction of extra explanatory factors (Svensson, 1995) or of dynamic factors (Diebold & Rudebusch, 2013). The introduction of dynamic factors is particularly interesting as it introduces time variation to the original model while having factors that carry some economical interpretation.

In the scope of Solvency II however the proposed methodology for the derivation of the risk-free yield curve is the Smith-Wilson framework. Moreover EIOPA (2014) suggests that the UFR be set at 4.2%, based on the assumption of 2.2% long-term economic growth and 2% inflation, with an LLP of 20 years and a convergence period to the UFR of 40 years for the Euro area. The upcoming implementation of this proposal has renewed the interest in the yield curve especially in the countries where similar regulations have been implemented such as Sweden and the Netherlands. For example, Budiono (2012) tests the use of a variable UFR and a 30-year LLP on pension fund performance, Chang and Li (2011) attempt extrapolation with a volatility term structure, and Rebel (2012) finds that the 20-year LLP provokes an unnatural behavior in the interest rate sensitivity of pension obligations.

1.3 Methodology

I have used Bloomberg to collect data on daily euro fixed-for-floating swap rates between August 15th 2001 and November 19th 2014 maturing between 1 and 30 years. My choice of the Euro swap rates is due to the fact that these new regulations will be applied within the Eurozone even though other countries (like other European countries and the US) are likely to face similar constraints in the near future. Having said that, my analysis could be easily extended to other regions and countries. I ran two estimations – one assuming an LLP of 20 years and a second one assuming data is available up to 30 years.

The two models being compared are the dynamic Nelson-Siegel model and the Smith-Wilson method. I chose the dynamic Nelson-Siegel (DNS) model (Diebold & Li, 2006; Diebold & Rudebusch, 2013) because as it is a time series model and not just cross sectional, it is a superior approach for forecasting over the original model as it describes dynamic properties of the yield curve. My approach will be on a first step to estimate the loading parameter λ from the data. Koopman, Mallee, and Van der Wel (2010) have found that introducing a time-varying loading parameter and volatility improves the fit of the model. I decided to see how the use of a time-variant factor loading λ would affect my results. The Smith-Wilson method uses the available data to exactly fit bond prices where data is available and to extrapolate them by using a weighted average of the last observable data point and the pre-determined UFR. The speed of convergence to this level is determined by the mean reversion parameter α, which can be compared to λ in the DNS.

1.4 Contribution

Although there are many empirical studies using the Nelson-Siegel model or extensions of it, the literature on ultra-long maturities is still scarce and as I have discussed this is a subject of relevance for some economic agents. Research tends to test such models with data on very liquid securities, such as bonds with maturities of between 3-months to 10-years, leaving open the question of whether or not longer termed securities could be used and if so when the last liquid point should be defined. This is especially important when we start dealing with the need to have a market-consistent discount curve that can value liabilities over 60 or 70 years. Thus, a study of the models in the literature with this time span in mind could actively help shape the decisions of these investors in the future. Furthermore, because interest rates are so important both in financial markets and the real economy, and bond markets are still developing at a rapid pace I expect this study could be helpful in shaping future research into yield curve modeling and forecasting.

2. Literature Review

As explained the main interest of the thesis is to do a comparison of term structure models currently available in the literature when they are used to extrapolate interest rate yield curves at ultra-long maturities. To clarify, the term structure of interest rates (or yield curve) shows the relationship between zero-coupon bond yields and their term to maturity. When plotted, its shape can give valuable insight, for example, about the expected path of future interest rates.

2.1 Traditional Term Structure Models

Before elaborating on the models that aim to model the term structure I will start out by giving a brief overview of traditional yield curve theories. These are the basis for all attempts to model the yield curve and in particular I find it important to mention the motivations behind the two main streams of literature on traditional yield curve models, the Expectations Hypothesis and the Liquidity Preference Hypothesis, as they will help understand later on the results obtained. One of the most popular term structure models in traditional literature is the Expectations Hypothesis having been developed over most of the 1900's, starting with Fisher's proposition that investors' expectations of the future spot rates affected current long rates. While this theory can't be pinpointed to one individual, Modigliani and Shiller (1973) provide a clear description: " [it] hypothesizes that in a world in which the future short-term rates are not known with certainty, the current yield of an *n*-period bond can be expressed as the very same function of the short rates currently "expected" to rule over the next *n* periods".

This hypothesis clearly has some limitations. Cox et al. (1981) state that it seems to be an adaptation of results in a certainty world to an uncertain reality as they show that the theory in its many formulations fails to hold under situations where interest rates are stochastic. They test four interpretations to the theory: the local expectations, the return to maturity, the yield to maturity and the unbiased expectations where the latter two overlap. It seems most of these formulations are lacking due to failing to realize that even in the presence of risk neutrality bonds have a term or risk premium.

The first criticism in the literature came from Hicks (1946) who added to this proposition by introducing the concept of a risk-premium. He builds upon Keynes' Liquidity Preference hypothesis that rates are determined partially by default risk and partially, on the long term, by uncertainty of future interest rates. He agrees that there is such a premium and adds to the original Expectations Hypothesis by saying that the long-term rate will generally exceed the average of the expected future spot rates by this risk/liquidity premium.

This framework proposed by Hicks' – the so-called Liquidity Preference model – can be considered a special case of the Preferred Habitat theory (Cox et al., 1981), where it is hypothesized that all investors have a preferred maturity in which they want to invest in (for example due to availability of funds for that time frame) but that investors may be enticed to invest outside of their preferred habitat due to higher expected returns. Thus the Liquidity Preference model could be considered the case in which all investors have a habitat that coincides with the shortest holding period. Putting it in these terms they interpreted the investment habitat preference as a risk rather than a time preference – investors who are less risk averse will demand positive term premiums while those who are more risk averse are fine with a negative premium. This interpretation seems to be more close in hand with reality – investors will tend to invest based on their risk tolerance instead of time preference especially as bond markets are functional and allow for trading of securities (whether or not the market is liquid enough should be incorporated in the premium).

I found it also interesting to point out that using the Liquidity Preference model there is evidence supporting the fact that past real interest rates and rates of inflation were the main variables containing information about future rates (Modigliani & Shiller, 1973). In particular when plotting results it's possible to see that (a) the short-term rate's lag structure was more characterized by extrapolative tendencies implying rational expectations, and (b) when forecasting the very short-term rates (for next quarter) information on the current rate was more important while the longer-term rates weighed more heavily on past information up to some years back and less so on the current rate. Modigliani and Shiller (1973) also found that the real rate of interest was strongly associated with current inflation, and that inflation itself was mostly impacted by its own past levels, having a lag structure with a strong regressive component.

2.2 Current Yield Curve Forecast Models

Thus far I have presented the main theories that have shaped the later development of the literature on the term structure. They provide some insight on how we can expect the term structure to evolve and what variables may play a role in shaping future interest rates. These are the basis for most of the literature that came afterwards when trying to address the problem of modeling the term structure of interest rates. This problem can actually be split into two: one is the concern with fitting the yield curve to empirical data, and the other determining what will be the evolution of the curve over time.

Initially the focus of this literature was on affine term structure models, which give a closed form solution to bond pricing by using an affine relationship between the spot rate and the state variables. They started out by being single factor models, such as the seminal works of Vasicek, and Cox, Ingersoll and Ross. To illustrate, Cox, Ingersoll, and Ross (1985) derived a single factor model based on general equilibrium where the traditional theories are incorporated by considering expectations about future events, risk preferences, characteristics of other investment alternatives, as well as investors' specific preferences on consumption timing. It is assumed that there is one source of market risk and this drives the evolution of interest rates, which follow a diffusion process. This type of assumption, usually present on single factor models, can be simplistic because by having the spot interest rate as the only explanatory factor it is inherent that rate changes would be perfectly correlated along the curve. Furthermore, with a single factor it is unlikely that there would be enough explanatory power to extrapolate yields beyond the last liquid point, being the model more applicable to short-term rates. There are also some multi factor models still within the affine class – one work worth noting in is that of Duffie and Kan (1996) for the generalization of the many models of this type in the literature. Nevertheless, it has been shown that this class of models – affine term structure models – are not very good at out-of-sample estimation, and actually do not outperform a random walk (Duffee, 2002).

Another stream of literature was concerned with ensuring a no-arbitrage condition when modeling the term structure of interest rates. Hull and White (1990) and Heath, Jarrow, and Morton (1992) are important examples of such contributions, where the approach was cross-sectional instead of a time series, modeling the yield curve to fit perfectly at one specific point in time. While having an arbitrage-free model is interesting, the Hull-White model is concerned only with currently traded bonds thus lacking time dynamics and not offering a way to extrapolate the term structure unfortunately.

On the literature of term structures of interest rates there is also the Nelson-Siegel (henceforth NS) model which takes a more empirical rather than theoretical approach to yield curve modeling, being able to fit yield curves of diverse shapes (humped, S-shaped, inverted and monotonic). At its core it still has the Expectations Hypothesis from the idea that forward rates, as forecasts of spot rates, would be the solution to the differential equation that produces the spot rate. In the model proposed by Nelson and Siegel (1987) the spot rate for the maturity τ-periods ahead is given by

$$R(\tau) = \beta_0 + \beta_1 \left[\frac{1 - e^{-\left(\tau/_{\lambda}\right)}}{\frac{\tau}{\lambda}} \right] + \beta_2 \cdot \left[\frac{1 - e^{-\left(\tau/_{\lambda}\right)}}{\frac{\tau}{\lambda}} - e^{-\left(\tau/_{\lambda}\right)} \right]$$

Where λ, β_0, β_1 and β_2 are the parameters to be estimated that contain information on the level, slope and curvature of the curve. These parameters can be seen as having a different contribution depending on maturity – β_0 can be seen as the long-term component as it is a constant that does not decay to 0 in the limit, β_1 has a large contribution at short term maturities but decays rapidly to 0 given its exponential term, and β_2 when plotted against time to maturity has a shape similar to a bell, thus is more of a medium term driver. This model benefits from being parsimonious and from its simplicity, although it might under some circumstances not yield a perfect fit.

It's also important to note that whether or not the model fits perfectly the data is not necessarily correlated with it's forecasting ability, for example McCulloch (1971) develops a model using a cubic spline that can accurately fit the data but this type of process produces unstable forecasts as it diverges at long maturities (goes to infinity at the limit) hence severely misrepresenting flat curves. In the NS model the yield is bounded at β_0 when maturity is large and at (β_0 + β_1) for the instantaneous rate. Moreover, when defining λ (which when small has a better fit at low maturities due to rapid decay) we could choose to minimize the error term of each dataset but better results are obtained when this is set to fit across the entire sample without great loss of precision.

One important contribution to this model came from Svensson (1994), who adds an additional parameter to the original NS framework to improve fit especially when the data has irregularities (he studies Sweden's data in the period of 1992-1994). In light of this he includes a second curvature term, with two extra parameters, β_3 and λ_2, to provide a better fit in particular at the end of the curve where data is scarcer and hence where the NS model struggled to be flexible enough. The ECB uses the Svensson version of the model when fitting its own yield curves and the parameters are published daily on its website.

More recently the literature on yield curve modeling seems to have flourished from the original NS framework especially after Diebold and Li (2006) and Diebold, Rudebusch, and Boragan Aruoba (2006) dynamic factor model linking the original NS factors to macroeconomic variables thus giving them some intuitive interpretation. Concretely, Diebold et al. (2006) introduce variables such as inflation, the real economic activity and the monetary policy instrument. Diebold and Li (2006) test the model's (henceforth DL) ability to forecast and find that it performs well in-sample and out-of-sample for long maturities. Their 1-month-ahead forecasting results are disappointing but they find that it improves dramatically after the 6-month-ahead horizon. Moreover the DL model replicates the five stylized facts of the yield curve: the average yield curve (i) is increasing and concave; (ii) assumes a variety of shapes through time; (iii) has persistent yield dynamics but spread dynamics are much less so; (iv) typically has a short end more volatile than the long end; and (v) its long rates tend to be more persistent than its short rates. They base their approach on imposing structure based on simplicity and parsimony such that it enhances the out-of-sample forecasting ability of the model even if it means the in-sample fit will be slightly deteriorated for lack of model flexibility. Nevertheless, Pooter (2007) finds that when using a four-factor model based on Svensson's model or the NS extension of Björk and Christensen (1999), the out-of-sample results are satisfactory and the in-sample fit is better than the three factor model when looking at the root mean square prediction errors (RMSPE).

Koopman, Mallee, and Van der Wel (2010) take the DL dynamic model and introduce time-varying factor loadings and time-varying volatility. They find evidence that both extensions significantly improve the fit of the dynamic NS model (henceforth DNS). Furthermore they show that not only is the assumption of a constant λ not the most accurate, when it is used as a latent factor it is highly persistent and affects the dynamics of the slope and curvature factors.

One other interesting addition to the class of DNS models is the condition of no arbitrage put forward by Christensen, Diebold, and Rudebusch (2011), thus introducing the class of arbitrage-free NS models which are affine arbitrage-free term structure models with the DNS factor loading structure. This allows to bridge the gap between the class of affine models that is theoretically rich but empirically lacking, with the DNS which empirically works well but lacks in theory. Nevertheless it will not be very useful for this case as the adjustment made causes very long-term rates to converge to minus infinity due to the presence of a unit root in the level factor (Balter, Pelsser, & Schotman, 2014).

2.3 Developments on the Solvency II Proposal

Laying out these yield curve modeling frameworks is motivated by the recent debate on the technical aspects of the currently still under work new directive for pension funds and insurance companies operating in the European Union – the Solvency II proposal which is supposed to come into force in January 2016. The objective of the proposal is not only to harmonize these institutions across the European market but also, and more importantly, to enforce certain guidelines for minimum capital requirements (MCR) and solvency capital requirements (SCR). The technical debate comes in where there is a need to discount assets and liabilities to compute these requirements. This includes using market data when available and computing best estimates that accurately reflect the risk present when direct market data is not available (Steffen, 2008). The European Insurance and Occupational Pensions Authority (henceforth EIOPA) has conducted several field-testing exercises – Quantitative Impact Studies (QIS) – of Solvency II, the latest of which was QIS5 on insurance institutions (thus far occupational pensions have only been subject to one QIS in 2012).

The Committee of European Insurance and Occupational Pensions Supervisors (henceforth CEIOPS) issued a final letter of advice for the implementation of Solvency II Level 2 (CEIOPS, 2009) where they make some interesting considerations. First and foremost, in this document CEIOPS points out that the desired characteristics of the instrument in which the risk-free interest rate term structure is based on are (i) having no credit risk; (ii) being realistically achievable by all insurers; (iii) being reliable; (iv) being highly liquid for all maturities (closely related to its reliability); (v) having no technical supply/demand biases; and (vi) being available for all relevant currencies. CEIOPS notes the importance of the chosen extrapolation method in complying with the mentioned qualities, in particular realism, and producing sufficient financial stability once it will be used in discounting the value of liabilities that are due very far in the future. The further in the future the liabilities are, the larger the impact of the discount rate used on their present value, hence the importance of having some stability and adhering to the insurers' reality. In regards to credit risk they argue that government bonds (with triple-A rating) are safer than swap contracts posing less credit risk, and thus they should always be preferred as an instrument over the latter. This has changed after the sovereign bond market events that occurred after the publication of this letter – in the latest technical provisions issued by EIOPA, it is recommended the use of swap rates. Using the swap rate is particularly useful if there are inequalities within the same currency, as is the case within the euro area where the swap rate will be available for all member states more equally than government bonds, which can have quite big spreads among them (i.e. central vs. southern Europe sovereigns).

Some of these concerns were answered in the calibration documents for QIS5 (EIOPA, 2010b) where the Smith and Wilson (2001) method and Ultimate Forward Rate (henceforth UFR) approach were first proposed for extrapolation beyond the last liquid point (henceforth the LLP). In the previous field exercises the risk-free term structures had been provided. As of April 2014, EIOPA (2014) has disclosed the technical specifications for the preparatory phase of Solvency II implementation which detail the proposed methods for calculation of the risk free term structure, including the volatility adjustment, are detailed. In this document EIOPA defines the UFR to be "the percentage rate that the forward curve converges to at the pre-specified maturity" and, for a given currency, it incorporates the appropriate inflation rate and long-term economic growth rate. They assume that for most currencies the

UFR will be 4.2%, a result of 2% inflation rate and 2.2% long-term growth rate. Another important concept is that of the speed of convergence to the UFR, represented by the α parameter in the Smith-Wilson method and which also determines the smoothness of the curve. The higher the α the bigger the weight of the UFR (hence faster convergence), while a lower α gives more weight to the market data.

In terms of the use of the swap rate as the preferred instrument for the term structure derivation, it is also important to point out that it should be adjusted accordingly for credit risk (EIOPA, 2014). Specifically, there is credit risk embedded when the rate on the floating leg of the swap is determined which depends on the credit quality of the banks involved in the deal. The adjustment should be done as a fixed deduction across all maturities and in the amount of the spread between these rates and the overnight indexed swap rates of matching maturity.

Although modeling yield curves is a much debated topic, when it comes to extrapolation at maturities beyond those provided in financial markets for the purpose of Solvency II there is not much empirical work done on the matter. Throughout my research I have found that much of what has been written on the subject stems from the countries that have been early on adapters of a similar regulation, such as Sweden, the Netherlands, and Denmark.

Budiono (2012) proposes an alternative calibration to the 20 year LLP and 4.2% UFR by using a LLP of 30 years instead and a variable UFR based on market rates. She finds this would significantly improve pension fund performance in terms of funding ratio in periods of either high or low interest rates.

In an attempt to improve how required capital for interest rate risk was computed within the scope of Solvency II, van Beers and Elshof (2012) have used cubic Hermite splines to interpolate spot rates. They argue that in this way the term structure will better reflect the stress shocks implemented and the convergence to the UFR. When using this method each set of neighboring spot rates is interpolated by using a linear combination of Hermite functions, and so it is expected that it will not only assure the term structure smoothness but also that the term structure can assume realistic shapes. They find that this method would be particularly useful for complex insurance products, whereas simple liability portfolios like immediate annuities work relatively well under the simplistic assumptions proposed by EIOPA.

Faced with the problem of illiquidity beyond 10 years, little observable data and potentially spurious short-term rates, Chang and Li (2011) extrapolated a volatility term structure for Taiwanese data instead of the usual bond price term structure. By showing the decay in the volatility of long-term forward rates, deriving a volatility term structure can help determine the speed of convergence to the UFR. Their formulation used a GARCH and a T-GARCH to estimate volatility and then fitted the term structure based on Vasicek (1977), using optimization constraints consistent with EIOPA (2010b). While Liu (2008) derived the theoretical volatility by setting the forward rate from t-1 to t periods ahead to the proposed UFR, Chang and Li instead fitted their term structure directly for a set of parameters that allowed the curve to be extrapolated to the UFR. They found that using their proposed GARCH model the term structure would converge more rapidly to the UFR than by using the QIS 5 method. As the Smith-Wilson method fits observed data directly this could be a consequence of a low interest rate environment.

One of the pointers guiding my research was whether or not the maturity at which stability was achieved (T2) would be the same for the models I'm testing. EIOPA (2014) suggests the convergence period will depend on the country to which the method is being applied, suggesting 40 years for the Euro area with an LLP at 20 years. This means that after the 20 year mark, rates are a weighted average of the last observed forward rate and the pre-set UFR for a period of 40 years finally converging to the 4.2% UFR. In the DNS model, on the other hand, the curve will converge to the level parameter dependent on the decay parameter λ, which is estimated using the available data. In particular, following from Koopman et al. (2010) results, my guess would be that using the DNS model with a time variant decay parameter should be a better reflection of the available data than the assumed *T2*, thus my first proposal emerges:

Hypothesis 1. Using a decay parameter dependent of the observed data yields a better extrapolation than a pre-defined one.

Moreover, Rebel (2012) critics EIOPA's proposal by arguing that one of the flaws of the UFR Smith-Wilson methodology is its abrupt disregard for market data beyond the last liquid point of 20 years. Is there any evidence that liquidity declines significantly after the 20-year mark? Provided secondary market trading of both sovereign bonds and interest rate swaps is over-the-counter, it is not easy to answer this question. If trading volume is used as a loose proxy for liquidity we can see in figure 1 that, in fact, while there is a significant peak in trading volume around the ten-year mark (possibly due to market segmentation), the German bonds maturing in 30 years still have a fairly high weekly trading volume (about 130 000 millions USD). This is not by any means sufficient proof that the market data for such long maturities is liquid enough to be a reliable estimate of the true risk-free rate. But then neither is there any concrete proof for the 20-year last liquid point. At most we could argue that there is sufficient market liquidity up to ten years and then it significantly declines. We can then weigh what the trade-off would be between on the one hand using less data points but extremely reliable ones, or on the other hand, using more data points but without certainty that they are completely unbiased.

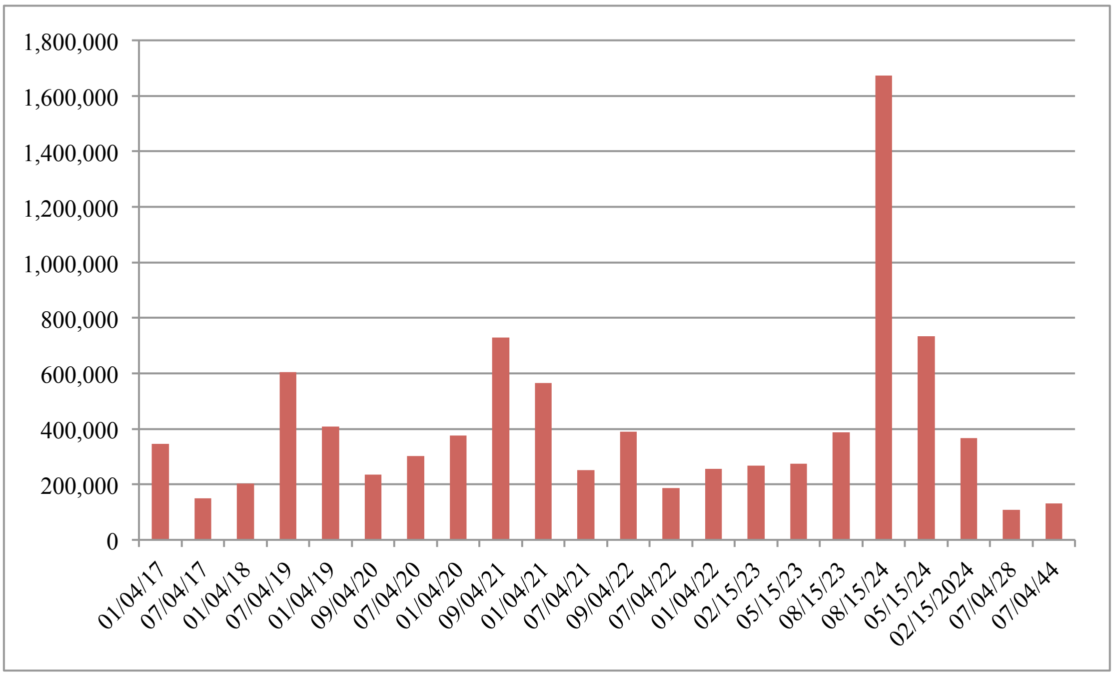

Figure 1: Trading volume in German Bunds' secondary market in the week 14/11/14 to 21/11/14, in millions USD. (Source: Bloomberg)

Hence I consider it would be of value to test whether there are significant differences on the extrapolation when using available data up to 30 years and propose the following:

Hypothesis 2. An LLP of 20 years is inadequate and using the available liquid market data up to 30 years improves extrapolation results.

My following question would be on how much the curves from the DNS model deviate from the fixed UFR on the long-end. Clearly, the level factor will predominate in the long run while the others will decay (Diebold & Rudebusch, 2013) making the curve stable. However it would be interesting to see whether the DNS curves have a significant amount of variation at the long end when compared to the constant UFR. This of course will depend on the decay parameter λ as it will determine the speed of convergence on the DNS model. Balter et al. (2014) extrapolate curves using the NS model (with an estimated λ set constant) and the SW method, finding that both provide a smooth curve beyond the LLP of 20 years. The SW curve lies above the NS due to the convergence to the UFR of 4.2%. I would like to test whether I find the same when using a time variant λ or not.

When attempting to compare both models another question arises: which model do we expect to provide a superior approach for the purpose of Solvency II? It is not clear. From what has been discussed thus far, I would expect the following:

Hypothesis 3. The SW-UFR model will provide a better fit in-sample and the DNS model a better out-of-sample extrapolation.

If this is true, how can we evaluate this trade-off? By using the available data, the DNS provides a more realistic discount curve on the long end, however it should be harsher for pension funds funding ratios when rates reach extremely low levels, as is the case now. On the other hand precision in fitting the curve at shorter maturities (i.e. before the set LLP) is not as good in the DNS as it is in the SW-UFR. As it has been previously discussed, pension funds have liabilities with maturities beyond the LLP (thus out-of-sample), so should my hypothesis be confirmed, it would be preferable to have a better out-of-sample fit at the expense of a slightly worse in-sample fit.

Finally, is it possible to say that the UFR set level of 4.2% is realistic? For one, it is arguably inconsistent with the current reality of interest rate markets. As it can be seen from figure 2, yields at the longer-term maturities have been significantly decreasing and even if we assume they will eventually mean revert it seems unlikely that it would be to a mean of 4.2%. The Netherlands UFR Committee (2013) recommends that the UFR is set on the average of the 20-year forward rates of the previous 120 months, which at the end of July 2013 would have yielded a UFR of 3.9%.

Figure 2: European Triple-A Government Bond Spot Yields, 20-year maturity. (Source: ECB Statistical Data Warehouse)

2.4 Model description

To research these issues I will be comparing the Dynamic Nelson-Siegel model (Diebold & Rudebusch, 2013) with the proposed SW-UFR method. The Smith-Wilson method is outlined in the Solvency II technical provisions (EIOPA, 2014). The price of a zero-coupon bond is defined as a function of coupon bonds as

$$P(\tau) = e^{-UFR*\tau} + \sum_{n=1}^{N} z_n * \sum_{t=1}^{T} c_{t,n} * W(\tau, \tau_t)$$

Where z_n is a set of parameters to be estimated with *N* observed data points maturity-wise, and $c_{t,n}$ stores the information on the cash-flows paid for a given *t* bond at a given n maturity. The W function, which can be seen as the equivalent to the loading matrix in the NS model, is defined as

$$W(\tau,\tau_t) = e^{-UFR*(\tau+\tau_t)} * \{\alpha * \min(\tau,\tau_t) - e^{-\alpha*\max(\tau,\tau_t)} * sinh[\alpha * \min(\tau,\tau_t)]\}$$

The variables τ and τ_t store the information on maturities across the whole data set. The first, τ, can be seen as the maturity on a given bond so it corresponds to the rows of the W function. The latter, τ_t, contains information on the maturities at which coupons are paid, so it maps the columns of W. Lastly, α is a pre-set mean reversion parameter that determines the speed of convergence. Due to the symmetry of the W function and being close to 0 when the maturity variables τ and τ_t are very large, it is clear that it will converge to the UFR. Moreover on the liquid part of the curve it will exactly fit the data available, while after that last liquid point, here determined by the chosen *N*, the yields will be constructed based on the data points available and the pre-determined continuously compounded UFR. In this model the idea is that the price function $P(\tau)$ is assessed as a linear combination of $\sum_{t=1}^{T} c_{t,n} * W(\tau,\tau_t)$, which can be compared to the NS model in the sense that it assesses the forward rate function as a sum of the three different parameters (level, slope and curvature).

It is also important to clarify that the yield curve, the forward rate curve and the discount curve are all related and thus easily interchangeable (Diebold & Rudebusch, 2013). The yield curve is related to the forward curve by

$$y(\tau) = \frac{1}{\tau}\int_0^\tau f(u)du \text{ with } f(\tau) = \frac{-P'(\tau)}{P(\tau)}$$

Where τ is the time to maturity, $y(\tau)$ is the continuously compounded yield and $P(\tau)$ the price of a discount bond maturing in τ periods.

The Dynamic Nelson-Siegel model as described in Diebold and Rudebusch (2013) is similar to the original model except that it introduces useful time dynamics to the parameters. This is achieved by considering instead of a cross sectional projection of $y(\tau)$, a time-series dimension for a given τ, where the parameters $\beta_{0,t}$, $\beta_{1,t}$ and $\beta_{2,t}$ become the variables.

$$y(\tau) = \beta_{0,t} + \beta_{1,t}\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_{2,t}\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right)$$

It relies on dynamic factor structure which is usually satisfied by financial data. As previously mentioned, the factors $\beta_{0,t}$, $\beta_{1,t}$ and $\beta_{2,t}$ have meaning – each will impact different maturities of the term structure, long, short and medium-term respectively – and can be seen as level, slope and curvature factor loadings.

The two models, the DNS and SW-UFR, are quite different in the way they are formulated. For extrapolation, on the one hand the SW-UFR method relies on the last known observation (at the LLP) and on the set UFR and the curve is created based on a weighted average of both for the period of convergence. On the other hand, the DNS method uses all the observed data to fit a curve and then uses the factor loadings to extrapolate the remainder of the curve beyond the LLP. But apart from this, both models have similar inputs. In the NS class of models we can read λ as being the decay parameter that determines the maturity at which the factor loading on β_2 (the curvature factor) is at its maximum. It can thus be seen as the *T2* used in the UFR model, as it marks the point at which the curve technically stabilizes. In DL λ was fixed at a predetermined value (at 16.42 a value such that the maximum would be reached at a 30-month maturity), whereas in Pooter it was estimated and then set constant for estimation. Moreover, while a UFR or specific constant level is specified in the DNS model, we know from its formulation that in the very long-run it will stabilize at β_1 so it can be seen as the UFR in the DNS.

So in summary, to address the issues identified, I will use the described models in 3 different ways: (i) by estimating the models as described, using an LLP of 20; (ii) using instead an LLP of 30 for both models; and (iii) by comparing the results obtained for DNS using a pre-determined λ and using an estimated λ.

3. Research Design

The outline of my methodology is as follows: I first started by interpolating the yield data available up to 30 years. As proposed in EIOPA (2014) the swap mid rate will be used for deriving both the DNS and SW-UFR curves. Then, after deriving the swap zero yield curve, I've fitted both the DNS and the SW-UFR models to that data and then followed each model's specifications in order to be able to extrapolate beyond the assumed last liquid point of 20 years. The DNS and SW-UFR will be calibrated in very different ways. In the case of the DNS I start with an initial estimation using a pre-determined λ and then I fit the data using maximum likelihood estimation and a Kalman filter. On the other hand for the Smith-Wilson method only simple matrices calculations are needed. I went through this procedure a second time using the available data up to the 30-year point. Finally I compared the results obtained.

3.1 Data

For the data collection I used Bloomberg and obtained the daily vanilla interest rate swaps fixed-for-floating Euro mid rates for the period of August 15th 2001 to November 19th 2014. This totals to 3453 observations. Although data for shorter termed swaps is available before August 15th 2001, this is the first observation I have for some of the longer maturities so in an attempt to harmonize the sample I chose this start date. For each date, rates were obtained for swaps with time to maturity of 1-year through 30-years, as well as swaps maturing in 40 and 50 years. The swaps in question have semi-annual settlement, tied to the six-month Euribor, and are quoted on a 30/360 day-count. The evolution of some of these rates over the sample period can be seen in figure 3.

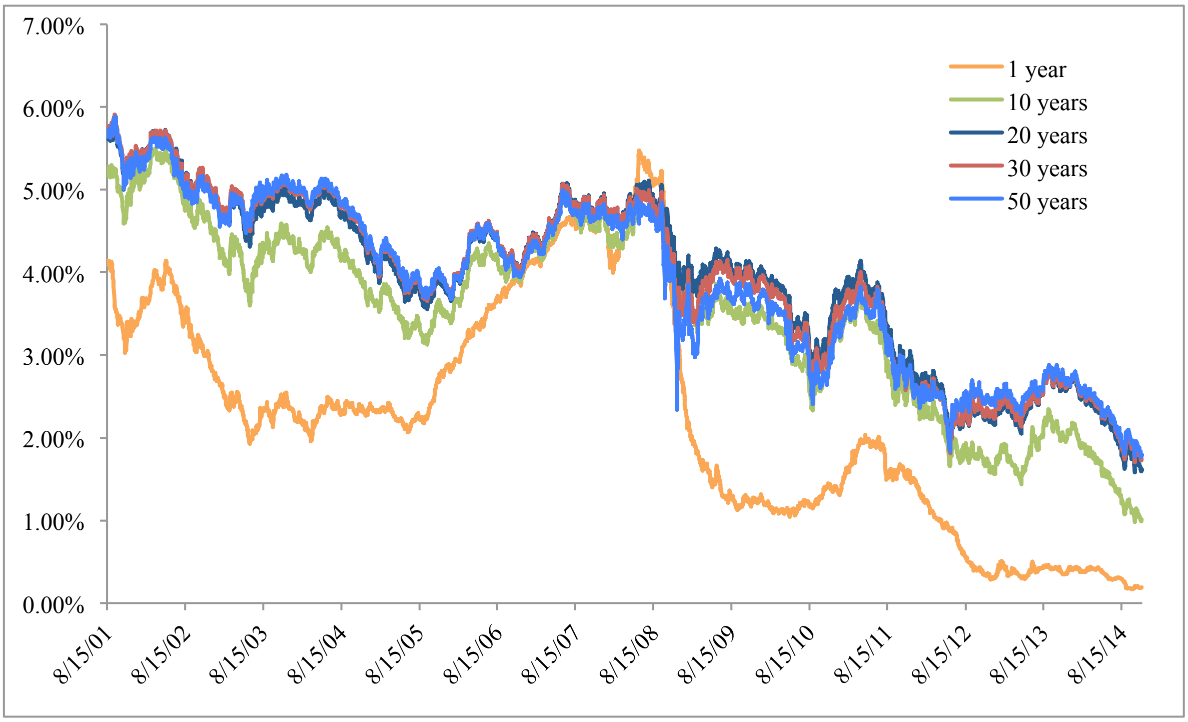

Figure 3: Evolution of fixed-for-floating Euro swap rates for swaps maturing in 1, 10, 20, 30 and 50 years. (Source: Bloomberg)

These swap rates are established as the midpoint between the bid and ask prices for contracts where the counterparties exchange fixed interest rate cash flows for floating cash flows, and vice versa. Thus at inception a swap does not involve any exchange of money, it is a zero net cash flow, and so it's value at that time is zero (Veronesi, 2010). This characteristic strongly reduces the risk involved in a transaction (particularly counterparty risk) and that is why swaps have become such an important instrument after the financial crisis of 2008. Knowing this the swap rate, s, can be expressed as

$$s = n \times \left(\frac{1 - Z(0, T_M)}{\sum_{j=1}^{M} Z(0, T_j)} \right)$$

Where $Z(0, T)$ are discount factors and there are cash-flow payments at times j through M, the maturity of the swap.

3.2 Determining the zero yield curve

As explained the first step in constructing the yield curve will be to bootstrap from the Euro swap mid-rates, the remaining maturities in the liquid part of the curve (Smith, 2012). However I am interested in the zero yield curve, so I have to transform the swap rates obtained, which for maturities over one year include cash flows (like a coupon bond), into zero yields. In order to do this, we look at the swap rates as if they were the coupon rate on a par bond in order to extract the unknown spot rates. The one-year spot rate, $R(0,1)$, will be the same as the one-year swap rate, s_1. For the following maturities, the continuously compounded $R(0,M)$ can be obtained from

$$100\% = \sum_{j=1}^{M} \frac{s_M}{e^{R(0,j)*j}} + \frac{s_M + 100\%}{e^{R(0,M)*M}}$$

In order to do this I used Matlab to extract the vector of $R(0,M)$ for each of the observation dates by finding first the discount factors and then transforming them to spot rates by the below relation.

$$R(t,T) = -\frac{ln\,(Z(t,T))}{T-t}$$

3.3 DNS model – constant loadings and time-variant loadings

Starting off with the parameterization of the DNS, there are several approaches that could be used for its estimation: a two-step process, a maximum likelihood estimation (using state-space representation and a Kalman filter), or even Bayesian analysis (in conjunction with Monte Carlo). While the two-step DNS approach by using linear regressions constitutes a simple and numerically stable method, it does not allow to calibrate λ based on the data available.

Thus I have decided to use the more precise maximum likelihood estimation (henceforth MLE) with a Kalman filter, also known as the one-step DNS, and since it can be sensitive to the initial parameters I have used prior to its estimation the two-step approach to establish a parameterization that is close to the one I'm looking for. For the initial two-step, or cross-sectional, approach I set lambda at 0.5 which is a good estimate when using yearly data. Then I performed a simple ordinary least squares estimation. The parameters obtained were then used for the starting values in the estimation of the state-space model. Following Diebold and Rudebusch (2013) the DNS yield curve can be transformed into a state-space representation of the model denoting the β parameters as level, slope and curvature to emphasize their meaning.

$$y_t(\tau) = l_t + s_t\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\right) + c_t\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right)$$

Then, the measurement equation can be defined by incorporating an error term, which stores information of any idiosyncratic movements in yields that are not driven by level, slope or curvature. This results in the following:

$$y_t = \Lambda f_t + \epsilon_t$$

$$\Leftrightarrow \begin{bmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \dots \\ y_t(\tau_N) \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{1 - e^{-\tau_1\lambda}}{\tau_1\lambda} & \dfrac{1 - e^{-\tau_1\lambda}}{\tau_1\lambda} - e^{-\tau_1\lambda} \\ 1 & \dfrac{1 - e^{-\tau_2\lambda}}{\tau_2\lambda} & \dfrac{1 - e^{-\tau_2\lambda}}{\tau_2\lambda} - e^{-\tau_2\lambda} \\ \dots & \dots & \dots \\ 1 & \dfrac{1 - e^{-\tau_N\lambda}}{\tau_N\lambda} & \dfrac{1 - e^{-\tau_N\lambda}}{\tau_N\lambda} - e^{-\tau_N\lambda} \end{bmatrix} \begin{bmatrix} l_t \\ s_t \\ c_t \end{bmatrix} + \begin{bmatrix} \epsilon_t(\tau_1) \\ \epsilon_t(\tau_2) \\ \dots \\ \epsilon_t(\tau_N) \end{bmatrix}$$

This means that in the liquid part of the curve a yield will be explained by level, slope and curvature factors depending on the parameters plus any other variation resulting from specific, or even "one-off", events. By fitting the available data in this way I obtain the parameters that will allow forecasting the yield curve beyond what is available. Then Diebold and Rudebusch (2013) specify the common factor dynamics by using a first-order vector autoregressive (VAR) process and defining the transition equation as

$$(f_t - \mu) = A(f_{t-1} - \mu) + \eta_t \Leftrightarrow \begin{bmatrix} l_t - \mu^l \\ s_t - \mu^s \\ c_t - \mu^c \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} l_{t-1} - \mu^l \\ s_{t-1} - \mu^s \\ c_{t-1} - \mu^c \end{bmatrix} + \begin{bmatrix} \eta_t^l \\ \eta_t^s \\ \eta_t^c \end{bmatrix}$$

Where μ is the factor mean vector and A the parameter matrix governing the factor dynamics. Moreover on the orthogonal white noise processes η_t and ϵ_t it is assumed

$$\begin{pmatrix} \eta_t \\ \epsilon_t \end{pmatrix} \sim WN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix} \right)$$

$$E(f_0 \eta_t') = 0 \quad and \quad E(f_0 \epsilon_t') = 0$$

The model is formulated in a way that the disturbances η_t are correlated, so the covariance matrix Q is non-diagonal, but the errors in observed yields, ϵ_t, are not correlated and the covariance matrix H is diagonal. In the described structure it is possible to use the Kalman filter for optimal extraction of the latent factors. The Kalman filter is an algorithm that provides a recursive solution to a least-squares problem by using a feedback process with equations that estimate and correct the estimate (Welch & Bishop, 1995). Hence this will help extract an optimal set of parameters from the state-space model described. I have used Matlab and the code provided in its website for the Diebold-Li estimation.

After obtaining the parameters, the yields are extrapolated by applying the formula below to all maturities τ up to 100 years.

$$y(\tau)_t^e = \hat{\beta}_{0,t} + \hat{\beta}_{1,t}\left(\frac{1 - e^{-\hat{\lambda}\tau}}{\hat{\lambda}\tau}\right) + \hat{\beta}_{2,t}\left(\frac{1 - e^{-\hat{\lambda}\tau}}{\hat{\lambda}\tau} - e^{-\hat{\lambda}\tau}\right)$$

3.4 Smith-Wilson method

As I am using a yield curve of zero coupon rates the equation for the SW method previously presented simplifies to the below form (Thomas & Maré, 2008).

$$P(\tau) = e^{-UFR*\tau} + \sum_{n=1}^{N} z_n * W(\tau, \tau_t)$$

As you can see, I can't just input the data I have obtained in the model – $P(\tau)$ yields a bond's market price. So in order to estimate this model, I have started by transforming the zero yields I had previously obtained into market prices of zero coupon bonds.

$$m_i = P(\tau_i) = e^{-\tau_i * \tilde{R}_{\tau_i}}$$

Then the rest of the model can subsequently be transformed into matrix notation by following the specifications provided by EIOPA (2010c).

$$m = p = \mu + W \cdot z$$

Where

$$m = (m_1, m_2, \dots, m_N)^T;$$

$$p = \left(P(\tau_1), P(\tau_2), \dots, P(\tau_N)\right)^T;$$

$$\mu = (e^{-UFR*\tau_1}, e^{-UFR*\tau_2}, \dots, e^{-UFR*\tau_N})^T;$$

$$z = (z_1, z_2, \dots, z_N)^T; and$$

$$W = \left(W(\tau_i, \tau_t)\right)_{i=1,2,\dots,N; j=1,2,\dots,N}$$

Laying it out in this way we can see that the parameter vector z needed for estimation can be obtained by inverting the Wilson function matrix and multiplying it by the difference between the market prices p and the μ vector (the asymptotical term).

$$z = W^{-1}(p - \mu) = W^{-1}(m - \mu)$$

Then all that is left is plugging in the obtained values of z into the initial model equation presented at the beginning of this section to find out the zero coupon bond market prices for the required maturities. To figure out the yield that corresponds to that price, the typical transformation for continuously compounded rates applies:

$$\tilde{R}_\tau = \frac{1}{\tau} \ln \left(\frac{1}{P(\tau)} \right)$$

For estimation purposes I have applied to this framework the latest Solvency II implementation guidelines, in particular that N = 20, UFR = 4.2% and $\alpha = 0.1$ (EIOPA, 2014). On a second estimation I substituted N = 30. When extrapolating the yields beyond each of these two LLPs, first the prices were found from the below equations and then transformed to yields by the relation presented above.

$$P_1^e(\tau) = e^{-4.2\% * \tau} + \sum_{n=1}^{20} z_n * W(\tau, \tau_n) \; and$$

$$P_2^e(\tau) = e^{-4.2\% * \tau} + \sum_{n=1}^{30} z_n * W(\tau, \tau_n)$$

3.5 Comparison

I have proposed that I compare both approaches of extrapolating the term structure so I have used some methods in order to evaluate the results I obtained. The first of such methods was to compute the Root-mean-square error (henceforth RMSE) in order to analyze the prediction errors, or in other words, to what extent the models' extrapolated yields may be off of the observed ones. The RMSE is defined as below where for a maturity τ of 40 or 50 years, $\hat{R}(0,\tau)_t$ are the extrapolated zero rates and $R(0,\tau)_t$ the observed zero rates at all data points t through M.

$$RMSE = \sqrt{\frac{\sum_{t=1}^{T}\left(\hat{R}(0,\tau)_t - R(0,\tau)_t\right)^2}{T}}$$

I have also used the RMSE to analyze the in-sample fit of the models for the observable maturities. In order to quantify the deviation in the yield curves long end I have looked at the mean curve estimated by each model and analyzed the difference between them. Finally, I have also performed simple visual comparison of the extrapolated yield curves.

4. Results

I will start out this section by presenting the descriptive statistics of the zero yields computed from the Euro swap rates. Then I will move on to showing the results from the model estimations, first from the Dynamic Nelson-Siegel and then from the Smith-Wilson method with the UFR.

4.1 "Zero-coupon" yields

As it can be seen in table 1, there is a very big spread in the data between minimums and maximums at all maturities. This was to be expected as over the past decade the trend has been for a very steep decrease in the interest rate level, especially after the financial turmoil of 2008. If the data were to be split into two subsamples, one prior to

Maturity (in years)	Max	Min	Mean	Median	Mode	Standard Deviation	Skewness	$\hat{\rho}_{30}$	$\hat{\rho}_{90}$	$\hat{\rho}_{180}$
1	5.33%	0.17%	2.24%	2.20%	0.41%	1.37%	0.2948	0.9695	0.8820	0.7234
2	5.38%	0.19%	2.39%	2.43%	2.72%	1.33%	0.0596	0.9639	0.8712	0.7218
3	5.26%	0.24%	2.58%	2.68%	3.09%	1.31%	-0.1329	0.9606	0.8666	0.7216
4	5.13%	0.31%	2.75%	2.89%	3.40%	1.27%	-0.2687	0.9580	0.8624	0.7185
5	5.06%	0.40%	2.92%	3.08%	3.64%	1.23%	-0.3622	0.9553	0.8580	0.7140
6	5.14%	0.51%	3.07%	3.21%	3.84%	1.19%	-0.4233	0.9531	0.8544	0.7105
7	5.24%	0.63%	3.20%	3.35%	4.01%	1.16%	-0.4611	0.9512	0.8518	0.7086
8	5.31%	0.75%	3.32%	3.47%	4.16%	1.13%	-0.4800	0.9498	0.8501	0.7082
9	5.38%	0.88%	3.43%	3.57%	4.28%	1.10%	-0.4881	0.9487	0.8487	0.7082
10	5.42%	0.99%	3.52%	3.66%	4.38%	1.08%	-0.4934	0.9475	0.8470	0.7074
11	5.48%	1.10%	3.61%	3.75%	4.47%	1.06%	-0.4943	0.9463	0.8451	0.7062
12	5.51%	1.19%	3.68%	3.84%	4.55%	1.04%	-0.4951	0.9451	0.8432	0.7049
13	5.56%	1.28%	3.75%	3.91%	4.61%	1.03%	-0.4923	0.9440	0.8412	0.7036
14	5.65%	1.35%	3.81%	3.96%	4.68%	1.02%	-0.4895	0.9433	0.8401	0.7034
15	5.71%	1.42%	3.85%	4.00%	4.75%	1.01%	-0.4869	0.9429	0.8396	0.7039
16	5.81%	1.47%	3.89%	4.04%	4.80%	1.01%	-0.4795	0.9424	0.8394	0.7053
17	5.87%	1.52%	3.92%	4.07%	4.85%	1.01%	-0.4718	0.9425	0.8400	0.7075
18	5.91%	1.57%	3.94%	4.09%	4.89%	1.01%	-0.4610	0.9426	0.8409	0.7100
19	5.94%	1.60%	3.96%	4.10%	4.93%	1.01%	-0.4475	0.9432	0.8422	0.7128
20	5.96%	1.64%	3.97%	4.11%	4.97%	1.02%	-0.4333	0.9439	0.8438	0.7159
21	6.00%	1.66%	3.98%	4.11%	4.99%	1.02%	-0.4183	0.9438	0.8440	0.7177
22	6.02%	1.68%	3.98%	4.10%	5.02%	1.02%	-0.3997	0.9440	0.8448	0.7199
23	6.04%	1.70%	3.98%	4.10%	5.04%	1.02%	-0.3788	0.9444	0.8454	0.7216
24	6.06%	1.72%	3.98%	4.08%	5.06%	1.03%	-0.3574	0.9445	0.8459	0.7229
25	6.06%	1.73%	3.97%	4.06%	5.07%	1.03%	-0.3378	0.9449	0.8468	0.7248
26	6.05%	1.74%	3.96%	4.05%	5.08%	1.03%	-0.3214	0.9453	0.8476	0.7263
27	6.05%	1.75%	3.95%	4.03%	5.08%	1.03%	-0.3063	0.9454	0.8482	0.7277
28	6.04%	1.75%	3.94%	4.02%	5.08%	1.04%	-0.2919	0.9457	0.8488	0.7291
29	6.03%	1.76%	3.93%	4.01%	5.09%	1.04%	-0.2773	0.9459	0.8494	0.7303
30	6.04%	1.76%	3.92%	4.00%	5.10%	1.04%	-0.2620	0.9461	0.8500	0.7311

Table 1: Descriptive statistics of the spot rates derived from the fixed-for-floating Euro swaps for maturities of 1 to 30 years. Autocorrelations for the 30-day (1-month), 90-day (3-months) and 180-day (6-months) lags are represented by $\hat{\rho}$.

September 15th 2008 (Lehman Brothers' filled for bankruptcy) and the other after, the differences are clearly noticeable especially on the shorter-end of the curve. In particular we can see from their autocorrelations how over the entire sample yields tend to be quite persistent, but this persistence dissipates a little after 2008 on the shorter maturities when we split the sample. Furthermore, while standard deviation of the yields is similar on the short-term in both sub-samples, it is 16 basis points higher for the 30-year yield on the post-Lehman sample. Also note how although maximums are quite similar in both subsamples the mean on the second subsample is much lower than that of the first subsample, reflecting the current low interest rate environment.

4.2 Dynamic Nelson-Siegel parameters

As mentioned, the estimation of the DNS model was made with both the full sample of maturities and with the smaller sample of 20 maturities, as suggested in the Solvency II implementation documents. In tables 2 and 3 below I present the descriptive statistics of the estimation of the level, slope and curvature factors with the dynamic model. It can be seen that the biggest change between the two estimations is the value of the decay parameter $\hat{\boldsymbol{\lambda}}_{\boldsymbol{t}}$, which comes closer to the fixed value of 0.5 with an LLP of 30 years. In terms of factor loadings, on average over the whole sample, the level is the only one that contributes positively to the yield. It tells us that in the very long-run when the short and medium term loading no longer matter that the yield would be approximately 4.58% (with an LLP of 20) or 4.52% (with an LLP of 30). Expectedly this is the most persistent factor of the three and has the smallest standard deviation. More surprisingly is the fact that the curvature factor, which can also be seen as the medium-term driver, is the least persistent of the three and when performing the estimation with the LLP of 30 its standard deviation significantly increases when compared to the other two factors. Looking just at the descriptive statistics for the factor loadings it would seem that the choice of last liquid point is not tremendously relevant. However when performing a significance test at a 95% confidence level (ttest2 function in Matlab), the null hypothesis that the factor loadings means are equal in both estimations is rejected for all three factors (with a t-statistic of 2.94, -8.15, and 17.77 respectively for level, slope and curvature).

I have estimated the two-step DNS for the initial parameterization of the state-space model, and it is worthwhile comparing the results obtained from both models to understand how using a data dependent λ may improve the estimation. I have included in annex the graphs that depict the evolution of the factor loadings estimated by the one-step and two-step DNS with both the 20 and 30-years LLP.

Factors	Mean	Standard Deviation	Min	Max	$\widehat{\boldsymbol{\rho}}_{\mathbf{30}}$	$\widehat{\boldsymbol{\rho}}_{\mathbf{90}}$	$\widehat{\boldsymbol{\rho}}_{\mathbf{180}}$
Level $\widehat{\boldsymbol{\beta}}_{\mathbf{1},\boldsymbol{t}}$	0.0458	0.0087	0.0240	0.0668	0.9227	0.8051	0.6823
Slope $\widehat{\boldsymbol{\beta}}_{\mathbf{2},\boldsymbol{t}}$	-0.0241	0.0126	-0.0466	0.0152	0.9528	0.8206	0.6169
Curvature $\widehat{\boldsymbol{\beta}}_{\mathbf{3},\boldsymbol{t}}$	-0.0228	0.0124	-0.0562	0.0013	0.8685	0.6649	0.4323
Decay Parameter $\widehat{\boldsymbol{\lambda}}_{\boldsymbol{t}}$				0.4268			

Table 2: Descriptive statistics of the DNS estimated parameters (level, slope and curvature) as well as the estimated decay rate for an LLP of 20 years.

Factors	Mean	Standard Deviation	Min	Max	$\widehat{\boldsymbol{\rho}}_{\mathbf{30}}$	$\widehat{\boldsymbol{\rho}}_{\mathbf{90}}$	$\widehat{\boldsymbol{\rho}}_{\mathbf{180}}$
Level $\widehat{\boldsymbol{\beta}}_{\mathbf{1},\boldsymbol{t}}$	0.0452	0.0089	0.0230	0.0662	0.9268	0.8121	0.6877
Slope $\widehat{\boldsymbol{\beta}}_{\mathbf{2},\boldsymbol{t}}$	-0.0217	0.0125	-0.0454	0.0170	0.9505	0.8149	0.6001
Curvature $\widehat{\boldsymbol{\beta}}_{\mathbf{3},\boldsymbol{t}}$	-0.0283	0.0137	-0.0650	-0.0005	0.8669	0.6616	0.4362
Decay Parameter $\widehat{\boldsymbol{\lambda}}_{\boldsymbol{t}}$				0.4930			

Table 3: Descriptive statistics of the DNS estimated parameters (level, slope and curvature) as well as the estimated decay rate for an LLP of 30 years.

It seems that the one-step estimation is much more controlled in high volatility times – as was the case in 2008 – but for the most part both estimation methods are quite comparable. As expected the long-term factor, the level loading, is decreasing over time. And if we look back at figure 3, it is possible to see how the 1-year rate (the shortest maturity depicted) resembles the slope loading, which portrays the short-term factor in the DNS. The most puzzling is again the curvature loading – graphically the estimation differs substantially using a different LLP. In particular the estimation using a longer LLP is more volatile, which is especially noticeable around 2008/2009 at the time of the financial crisis.

4.3 Smith-Wilson parameters

Once again the estimation of the Smith-Wilson parameters was done for both an LLP of 20 and 30 years. In this case the estimated parameters z do not have any economic interpretation as in the DNS as the model estimates parameters for each maturity that best fit that data point. This was also the reasoning behind proposing that this method would provide a better fit in-sample than the DNS. Taking this into account I will include the descriptive statistics for these parameters in annex just for reference but will not discuss them. Instead I will analyze the Smith-Wilson estimation in the next sub-section.

4.4 Estimated and extrapolated yields

Now I will present the yield curves resulting from the application of the calibrated models, including their extrapolation beyond the 20 or 30-years last liquid point and up to a maximum maturity of 100 years.

This analysis can be split into two: the in-sample fit (from maturities 1 to 20/30 years) and the out-of-sample fit (from maturities 21/31 to 100 years). Starting with the in sample fit it can be seen in figures 4 and 5 the average yield curve fitted using the Smith-Wilson and DNS models. The discrepancy when using 30 years of observable data becomes noticeable in the case of the DNS, which constructs a smooth curve by calibrating the parameters taking into account the entire data sample. Note also how it is almost impossible to distinguish the SW estimation from the actual zero rates – this is a result of the model's nature as it has been mentioned before, where the parameters' job is to achieve the best fit to the observable data and do not carry any economic meaning like in the DNS.

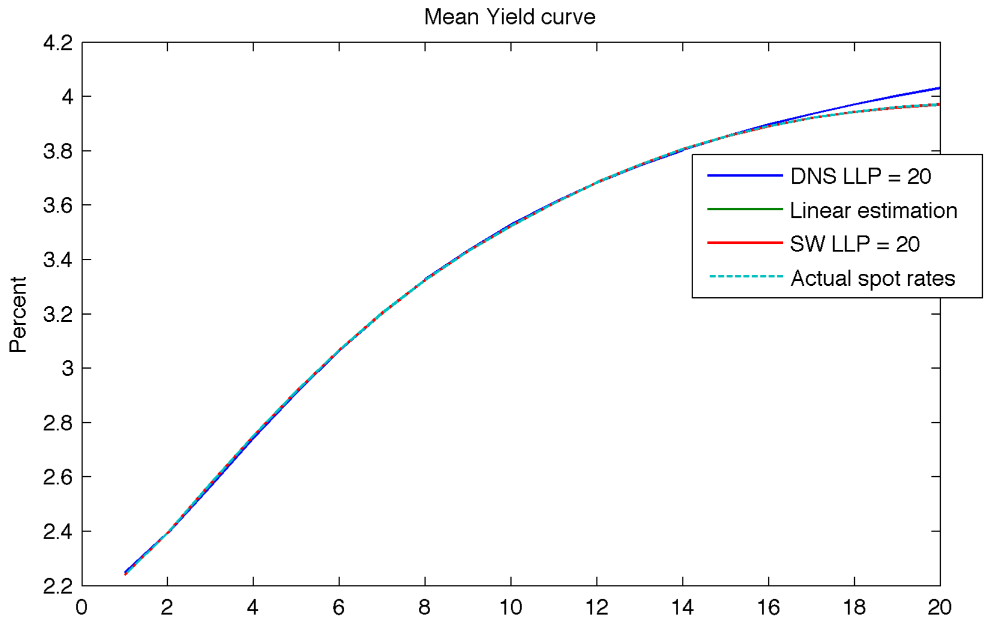

Figure 4: Mean yield curve fitted by the DNS and SW using an LLP of 20 years versus the actual spot rates and a linear interpolation method.

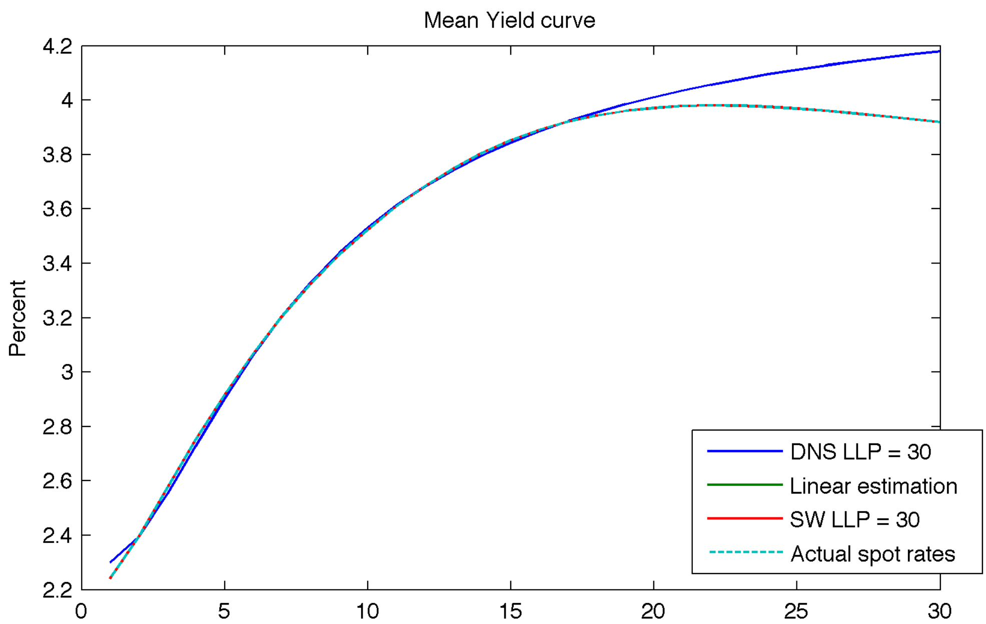

Figure 5: Mean yield curve fitted by the DNS and SW using an LLP of 30 years versus the actual spot rates and a linear interpolation method.

To quantify this deviation I have computed the RMSE for the in-sample maturities. The RMSE results confirm what is seen graphically above by showing that the DNS estimation deviates more from the actual spot rates than the SW estimation and that this difference worsens at the short and long ends of the curve. In the graphs provided in annex it is also possible to see that in distressed times, such as the day in which Lehman Brothers' filled for bankruptcy, the discrepancy between the market rates and the DNS estimation is bigger. It should also be noted that the yield curve on this particular day has a shape far from what would be considered a typical yield curve.

Maturity (in years)	SW20 ($\times 10^4$)	DNS20 ($\times 10^4$)	SW30 ($\times 10^4$)	DNS30 ($\times 10^4$)
1	0,0976	2,6793	0,0797	2,5706
2	0,1335	0,0046	0,1090	0,0038
3	0,1589	0,7383	0,1298	0,7537
4	0,1640	0,7478	0,1339	0,7631
5	0,1714	0,4983	0,1400	0,5081
6	0,1759	0,2421	0,1436	0,2363
7	0,1765	0,0636	0,1441	0,0391
8	0,1778	0,1715	0,1452	0,1658
9	0,1786	0,2366	0,1459	0,2333
10	0,1763	0,2667	0,1440	0,2641
11	0,1787	0,2210	0,1459	0,2445
12	0,1779	0,1600	0,1452	0,2234
13	0,1802	0,1233	0,1471	0,2304
14	0,1788	0,1589	0,1460	0,2243
15	0,1790	0,3279	0,1461	0,2367
16	0,1790	0,5705	0,1461	0,3065
17	0,1815	0,8932	0,1482	0,4785
18	0,1825	1,2934	0,1490	0,7399
19	0,1778	1,7596	0,1452	1,0698
20	0,1817	2,2998	0,1483	1,4672
21	-	-	0,3634	1,8488
22	-	-	0,5687	2,3024
23	-	-	0,8478	2,7787
24	-	-	1,2127	3,2807
25	-	-	1,5724	3,7768
26	-	-	2,0030	4,2668
27	-	-	2,4436	4,7599
28	-	-	2,8961	5,2416
29	-	-	3,3500	5,7079
30	-	-	3,7790	6,1506

Table 4: Root Mean Square Error for in-sample comparison of the two models for both LLPs considered.

When it comes to extrapolation beyond the last liquid point, the graphical results of the average extrapolated yield curve can be seen in figure 6 and the descriptive statistics of the extrapolated rates will be included in annex.

From the graphical representation two interesting things can be observed: the first is that surprisingly the DNS stabilizes at a higher level than the SW, which converges to the UFR as expected; the second is that using an LLP of 20 years in the SW method achieves a smoother transition, on average.

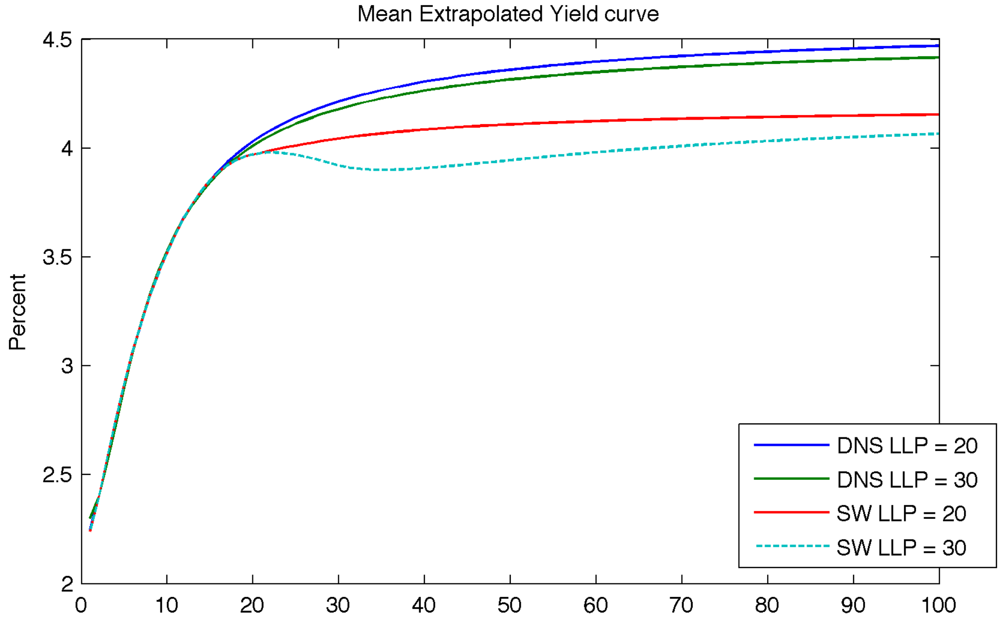

Figure 6: Mean yield curve extrapolated up to 100 years with the two methods of estimation, DNS and SW.

Regarding the first observation it can be explained due to the high interest rate level at the beginning of the sample. When splitting the sample as previously it is possible to see how that is the case since on the post-Lehman bankruptcy sample the long-end of the extrapolated curves is much more close together than in the first part of the sample, where the SW method deviates quite a bit due to the condition of convergence to the UFR. In terms of the second observation it indicates that the observed yield curves, on average, tend to have a downward inflexion after the 20-year period. Again this is particularly noticeable in the second part of the sample as can be seen in figure 7.

RMSE	DNS (LLP 20)	DNS (LLP 30)	SW (LLP 20)	SW (LLP 30)
40-years maturity	0,01691	0,01669	0,00283	0,00092
50-years maturity	0,01690	0,01670	0,00364	0,00183

Table 5: Root Mean Square Error for out-of-sample comparison of the two models at maturities of 40 and 50 years.

To quantify the out-of-sample fit of the models I computed the RMSE using the entire sample for maturities of 40 and 50 years for which I had the market data available but did not use for estimation purposes in either alternative of the LLP. The results show that the SW method deviates much less from the observed market rate than the DNS.

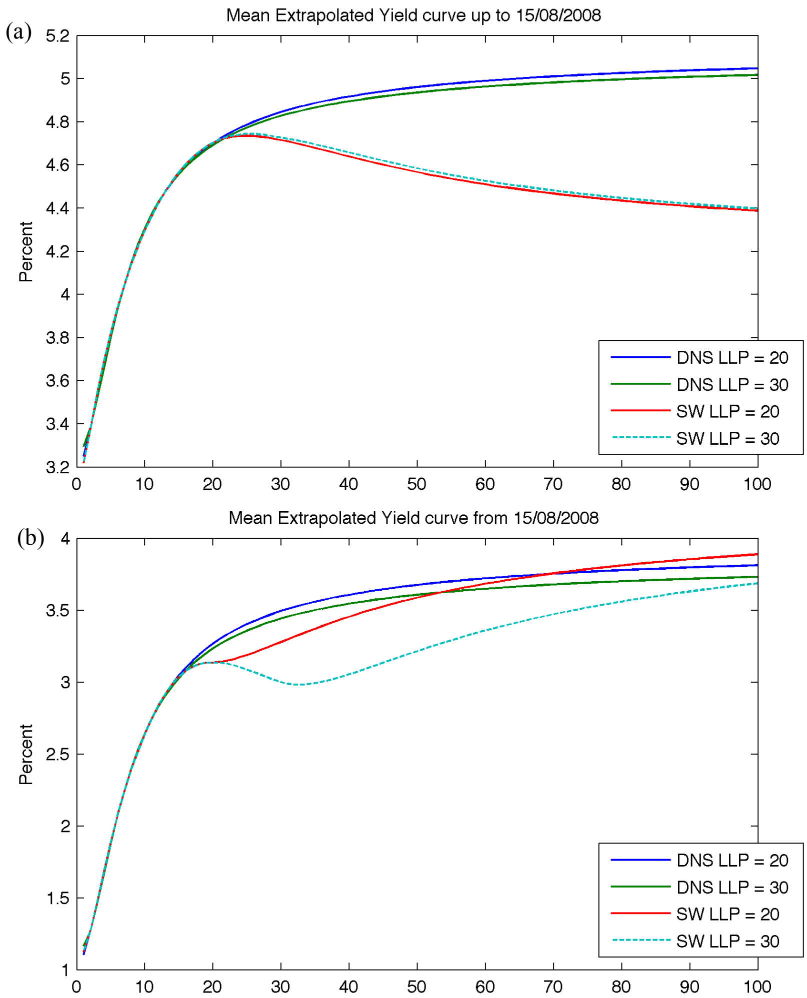

Figure 7: Comparison of the average yield curve extrapolated by two methods of estimation, DNS and SW, on (a) the pre-Lehman failure period and (b) the post-Lehman failure period.

This was already the conclusion drawn from the graphical representation. Furthermore it is possible to see that using a longer LLP of 30 years translates into a lower RMSE for both models, with the difference being more noticeable in the SW estimation.

5. Discussion

Now that I have exposed the results I have obtained from my research I will discuss their implications in relation to my initial research questions. Moreover I have also gained some insight as to whether or not the hypotheses I put forward earlier were reasonable. Refreshing what my initial hypotheses had been we have:

- **Hypothesis 1.** Using a decay parameter dependent of the observed data yields a better extrapolation than a pre-defined one.
- **Hypothesis 2.** An LLP of 20 years is inadequate and using the available liquid market data up to 30 years improves extrapolation results.
- **Hypothesis 3.** The SW-UFR model will provide a better fit in-sample and the DNS model a better out-of-sample extrapolation.

One of the things I was looking for was to see whether both the DNS and the SW estimated curves would achieve stability at the same maturity. In the computation of the SW method, the maturity *T2* should be around 60 years using an LLP of 20 years (70 for an LLP of 30 years). Looking at the descriptive statistics of the average extrapolated yield curves it is possible to see that in the case of the SW stability is indeed achieve sooner in the case where the LLP is shorter. Because the curves never really reach the UFR it is not possible to say that they definitely stabilize after the 40-year convergence period. However they seem to be relatively stable at the 45-year mark and 65-year mark for an LLP of 20 and 30, respectively. In the case of the DNS stability seems to be achieved around the 60-year mark independently of which LLP was used for estimation. Although both DNS curves achieve the UFR of 4.2% and surpass it, as it has been mentioned this model converges on the very long term to the factor loading β_1, so I have used instead as a gauge for stability rates that were constant for four years in a row. Achieving stability is closely linked with the decay parameter λ and thus with my first hypothesis where I had initially proposed that calibrating the decay parameter λ with the available data instead of using the pre-determined 0.5 value would produce a better estimation of reality. I tested this hypothesis by analyzing both the one-step and two-step methods of estimation of the DNS. In terms of the factor loadings, as I have mentioned in the previous section, the difference albeit not big, was noticeable particularly in periods of high turbulence in financial markets. This shows that my initial hypothesis was correct. To reinforce this, in figure 8 it is possible to see that the one-step method of estimation (where λ is estimated) fits the actual rates much more closely than the two-step method. Since most the 2008 financial crisis financial markets have had a more erratic behavior, especially European markets, it seems that the estimation of the decay parameter is valuable as it will more accurately capture the observed yield curve and provide for a more realistic discount curve. Comparing the DNS and SW, the use of an estimated λ instead of a defined *T2* produces differences in terms of achieving stability as well – in the SW it depends on the chosen LLP while in the DNS it does not matter as stability is achieved at similar maturity for both LLPs. By analyzing the inputs that go into the models' estimation, I naturally come to one of the central questions of this paper, which is whether or not using an LLP of 30 years is more beneficial than the proposed 20 years. Regarding this second hypothesis I made, the conclusion was not as straightforward as I thought it would be. Graphically we can see that using a longer LLP makes the average extrapolated curve have a more pronounced downward inflexion before it starts converging to the UFR. This behavior is especially motivated by the second part of the sample (post-financial crisis) as it can be seen from figure 7. This produces a strange looking yield curve but does not imply anything about whether using a longer LLP is better or worse when producing an accurate risk-free curve. When we look at the RMSE for 40 and 50 years we can see that the use of a longer LLP reduces the error for both the DNS and the SW models, which shows that the extra information is in fact useful for extrapolation. Of course one could still argue that there are liquidity constraints at these longer maturities and that they do not exactly reflect what a risk-free yield at these maturities would be priced at. Given my earlier discussion there should not be much reason for concern on the liquidity front nevertheless if one believes that problem exists it is always possible to include an illiquidity premium in the estimation. In this case I have chosen not to do it as I do not think that problem existed. Also note how the inflection of the extrapolated yield

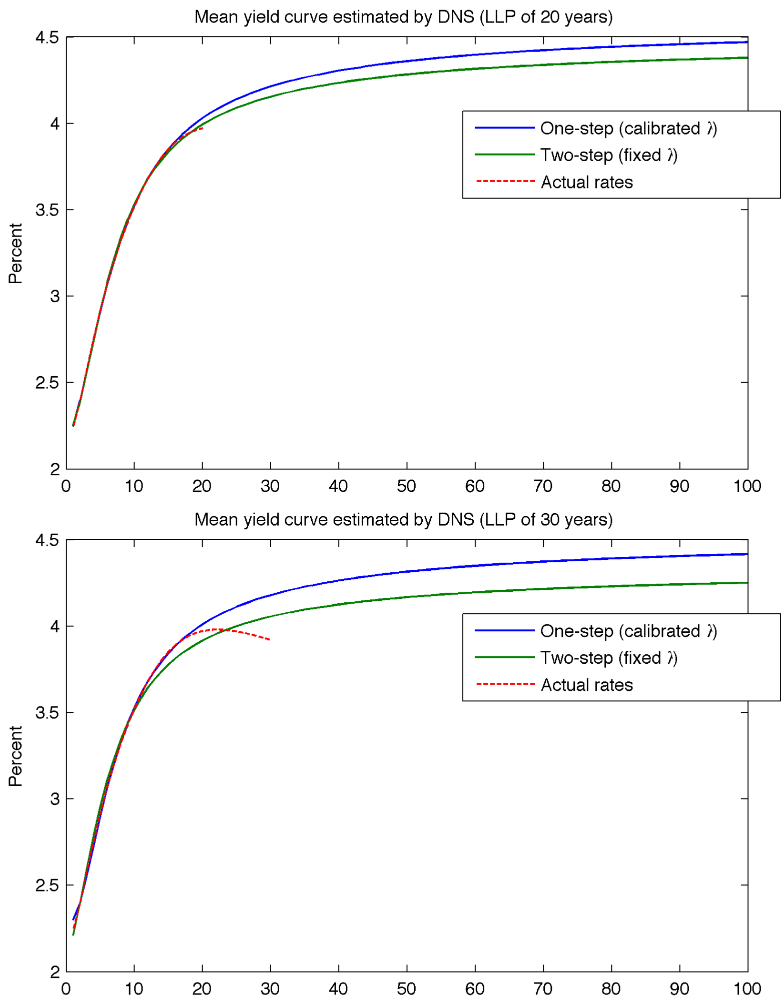

Figure 8: Comparison of the average yield curve extrapolated by the DNS using a pre-determined and a calibrated λ

curve becomes more pronounced in the second part of the sample using an LLP of 30 years – I would attribute this behavior not to an eventual lack of liquidity in the rates but to an inadequacy of the selected UFR. By using the longer data set of 30 maturities, more information is incorporated about the future rates and thus results in a more accurate risk-free discount curve without any significant loss due to liquidity constraints. This brings me to the comparison of the two models' curves at the long end. As it can be seen in figure 6 the DNS curves on average attain a higher level on the long end than the SW curves. When splitting the sample it is evident that the level at which the DNS curve stabilizes will highly depend on the general interest rate level at the date of estimation, whereas the SW always converges to the same UFR level. So in this case because the sample spans a very long period of time that comprises very different interest rate environments, the DNS appears on average above the SW. However if we look at the bottom half of figure 7 it is possible to see that at the long-end the two models do not differ that much, with the exception of the SW estimated with 30 years of data. It is possible to say that the DNS is much more data reliant when extrapolating than the SW which when looking at the yield curve extrapolated for September 15th 2008 (in annex), seems to almost disregard its previous trend in order to converge to the pre-determined UFR. Also worth noting is that if we extrapolated the yield curve for the most recent data point in my dataset, the SW curves already lie significantly above the DNS ones, and both at the longest maturity are still quite far from the proposed UFR.

Keeping with the subject of the proposed 4.2% level for this rate, there seems some mismatch between the level at which it is set at the moment and the level of current market rates and the extrapolated DNS curves. This is reinforced by the fact that neither model actually achieves the proposed UFR of 4.2%, not only in the last data point of the sample but also not in the average curve of the second half of the sample. In fact, taking November 19th 2014 as an example, even by calibrating the α parameter in order to have the SW model converge as close as possible to the proposed UFR, after 100 years that level would still not be reached. By calibrating the alpha for this day I obtain for an LLP of 20 years an α of 0.146, and for an LLP of 30 years an α of 0.514, unlike the proposed 0.1.

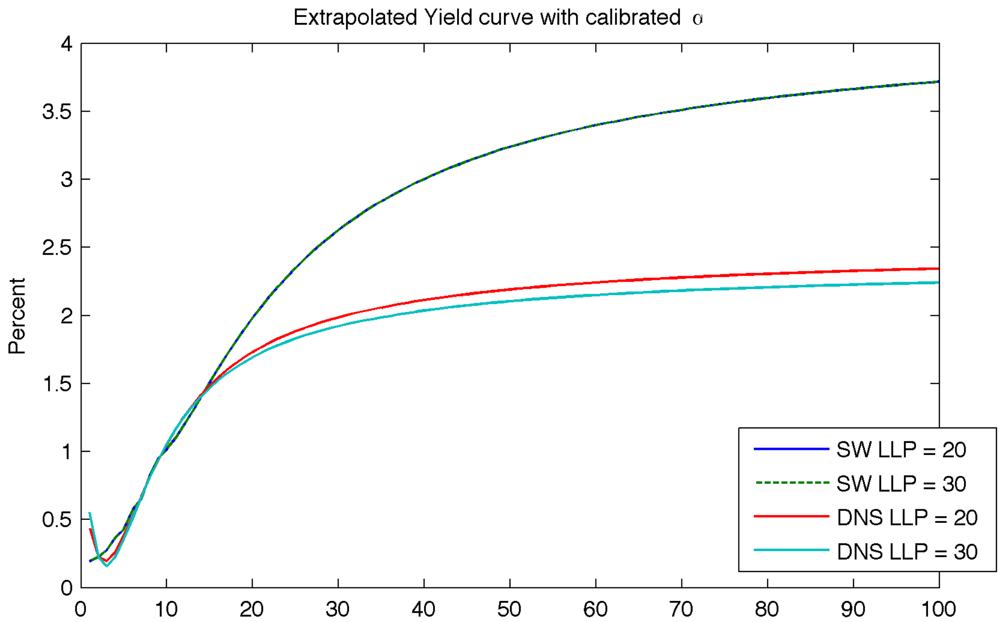

Figure 9: Extrapolated yield curve on November 19th 2014 after calibrating α for the SW method, versus the estimated DNS.

The fact that this UFR level may not be realistic is also supported by very low inflation levels in Europe in the past years, as well as slow economic growth and a not very good growth outlook for the near future. As this is supposed to be a very long-termed interest rate level it can be understood why EIOPA (2014) would set it at this higher level but as the lower rate trend continues it might be worthwhile to revise it in order to avoid struggling pension funds and insurance companies in the future.

Finally, following my literature review I had proposed that when comparing the two models I would find that the SW method would present a better in-sample fit than the DNS and that out-of-sample the DNS would outperform the SW. By computing the

RMSE I was able to verify that the SW does generally fit better in-sample than the DNS, with the DNS deviating more from the actual observations on the shorter and longer maturities of the curve but being very close to the SW on the medium term. Despite the first part of my hypothesis being confirmed, I found that the DNS did not outperform the SW on the out-of-sample fit – the RMSE for maturities of 40 and 50 years was significantly larger for the DNS model than for the SW method. I believe this is due to the previously mentioned downward trend at the end of the observed curve and the lack of flexibility of the DNS to adapt at the long end. It is possible that this could be mitigated by an additional fourth factor as in Svensson (1995) which would better capture this movement at the end of the observed data.

Overall which model should we expect to provide the superior approach? As we have seen the DNS is the most market consistent at the long end, even if the RMSE for the 40 and 50 maturities is worse than that of the SW it is clear that graphically it is more consistent with the prevalent data, particularly in financially distressed periods. For the purpose of Solvency II however this might not be ideal, as it would imply a high degree of uncertainty and instability in the valuation of liabilities.

6. Conclusion

The purpose of this paper was to cast some insight into two popular term structure extrapolation models applied in the scope of the soon to be implemented Solvency II regulation. One of these models, the Smith-Wilson, is the one provided as part of the technical provisions for the implementation of this regulation. The other is the Dynamic Nelson-Siegel model, an extension of the original Nelson-Siegel model that incorporates economic meaning into its parameters. I was mostly interested in three aspects: the number of data points used for estimation (the last liquid point), the rate at which the curve stabilizes in the long-run (the UFR), and the time it takes the curve to converge to that level (the convergence period T2).

I found that using a last liquid point of 30 years instead of 20 years does improve both the in-sample and out-of-sample fit of both models. Nevertheless this result should be taken lightly as the spot rates used show a strange behavior after the 20-year maturity. This is probably due to market segmentation: the same institutions that benefit from this study, pension funds and insurance companies, will have a strong demand for swaps at the longest maturities available but there aren't as many counterparties available as for shorter maturities. This problem could be solved perhaps by using a different instrument that was not as exposed to this problem but that would also imply using a less liquid instrument and thus not truly producing a risk-free discount curve. It still makes sense to incorporate as much information available as possible so my advice would be to still use the available data up to the 30-year LLP. It is also important to remember that this erratic behavior between the 20 and 30-year maturities was exacerbated during the financial crisis, so in normal times there is no reason to shy away from this data. Even if there is some concern about the reliability of the data there is always the possibility to input an illiquidity premium in the estimation.

In terms of the convergence period, or in the case of the DNS the decay rate, as expected after my literature review I found that using a calibrated λ using the available data improves the estimation of the yield curve versus a pre-determined λ. This resulted in a more uniform convergence period – the curve stabilized around the same maturity independently of being estimated with a 20-year or 30-year LLP – whereas in the SW it varies with the LLP due to being fixed at the proposed level, which for the Euro area is a convergence period of 40 years and a parameter α of 0.1. When calibrating this parameter for the last data point only two things could be noted: the first was that the estimation for the two different LLP matched, and the second was that the convergence period using the 30-year LLP jumped to 0.514 instead of the proposed 0.1.

One other insight that could be drawn from this exercise was that even by calibrating the α, the SW did not reach the proposed UFR at the longest maturity. In fact if we take a section of the sample from September 15th 2008 to November 19th 2014, on average, none of the estimated models achieved a level above 4% before the last extrapolated maturity of 100 years. Furthermore, if we look at the extrapolated curve for the day Lehman Brothers' filled for bankruptcy, the discrepancy between the DNS and SW estimation is huge, almost of 1% at the longest maturity. This is due to the convergence to the pre-set UFR. For a pension fund to see its very long termed liabilities be discounted at a rate that is wildly different than the one it rationally extracts from the market (in this case almost one percent below) could put the pension fund at jeopardy. Setting a UFR basically removes an important component of having a market consistent approach by pre-determining the rate. On the other hand, it is also helpful in financially distressed times – it helps the liabilities' value to be protected from big interest rate swings. So how can we reconcile these two extremes? A good solution would be to regularly update the UFR taking into account the current interest rate environment along with economic growth and inflation.

In the end, ideally we would like to have the market consistent and economically interesting features of the DNS, but also the possibility of putting a cap on the how much that would impact the liabilities' valuation when for example financial markets are distressed and don't accurately reflect the longer term reality. My advice would be to still use the DNS to extrapolate a risk-free yield curve by using an average of the previous year's curves for example – this solution should provide a market consistent approach and the possibility to interpret the factor loadings estimated while still protecting the pension funds and insurance companies from big swings in rates. Moreover, if some of the longer rates are not 100% reliable, as it may be the case in this sample during the financial crisis (due to the rates inexplicably dropping between 20 and 30 years), the DNS is more likely to extrapolate the curve in a more even manner than the SW method. The latter by exactly fitting the available data takes the full impact of any erratic behavior the rates might have. Of course the use of the DNS should be complemented by EIOPA's regular expert advice to these institutions in how to best comply with the new Solvency II regulation.

Finally, I believe there is still some room for improvement in my analysis, especially if the DNS model is to be applied in the scope of Solvency II. For example it would be interesting to apply Koopman et al. (2010) time-variant decay parameter in the estimation as they argue that fixing λ even after calibration may be too restrictive of a condition, especially when the sample spans a long period of time as is the case here. Also it would be interesting to research whether the inclusion of a fourth factor would produce a better fit for the DNS model and whether that produces a more accurate curve or not. This should be done along side a test for whether or not choosing a sample that includes the financial crisis years influences spot rates between 20 and 30 years like I believe it has in my research. On the one hand, if the downward inflection in the data is not caused by the financially disturbed years then either a different instrument should be chosen or we have to settle for a 20-year LLP despite its drawbacks. On the other hand if it is a problem caused by the disturbances during the financial crisis years in the sample, it might be a good thing that the DNS does not fit the data as well as the SW. The SW ends up producing a discount curve that fits well the erratic data but in the long-term will prove to be inaccurate. This discussion may not be extremely relevant for the very long-termed maturities at the end of the extrapolated curve but as many pension funds and insurance companies have liabilities maturing right after the last liquid point (around 40 to 50 years) it would be important to further research this issue within the scope of Solvency II.

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Annexes

Maturity (in years)	Max	Min	Mean	Median	Mode	Standard Deviation	Skewness	$\hat{\rho}_{30}$	$\hat{\rho}_{90}$	$\hat{\rho}_{180}$
1	5.33%	1.91%	3.22%	3.14%	2.30%	0.92%	0.3956	0.9405	0.8035	0.6549
2	5.38%	1.99%	3.38%	3.35%	2.72%	0.80%	0.2420	0.9215	0.7369	0.5676
3	5.26%	2.22%	3.55%	3.57%	3.09%	0.71%	0.1526	0.9057	0.6966	0.4962
4	5.13%	2.45%	3.69%	3.72%	3.40%	0.65%	0.0823	0.8948	0.6742	0.4442
5	5.06%	2.60%	3.82%	3.81%	3.64%	0.60%	0.0284	0.8884	0.6672	0.4130
6	5.14%	2.73%	3.94%	3.92%	3.84%	0.57%	-0.0019	0.8856	0.6701	0.3982
7	5.24%	2.84%	4.05%	4.03%	4.01%	0.55%	-0.0144	0.8851	0.6788	0.3979
8	5.31%	2.95%	4.14%	4.13%	4.16%	0.53%	-0.0207	0.8856	0.6895	0.4069
9	5.38%	3.04%	4.23%	4.22%	4.28%	0.52%	-0.0273	0.8865	0.6997	0.4208
10	5.42%	3.13%	4.30%	4.31%	4.38%	0.52%	-0.0386	0.8878	0.7086	0.4352
11	5.48%	3.21%	4.36%	4.39%	4.47%	0.51%	-0.0435	0.8887	0.7167	0.4497
12	5.51%	3.28%	4.42%	4.47%	4.55%	0.51%	-0.0522	0.8900	0.7240	0.4636
13	5.56%	3.33%	4.48%	4.55%	4.61%	0.51%	-0.0615	0.8912	0.7298	0.4758
14	5.65%	3.39%	4.52%	4.60%	4.68%	0.51%	-0.0732	0.8920	0.7347	0.4870
15	5.71%	3.44%	4.57%	4.64%	4.75%	0.51%	-0.0813	0.8927	0.7385	0.4954
16	5.81%	3.49%	4.60%	4.67%	4.80%	0.51%	-0.0865	0.8915	0.7404	0.5024
17	5.87%	3.53%	4.63%	4.71%	4.85%	0.51%	-0.0936	0.8914	0.7427	0.5085
18	5.91%	3.56%	4.66%	4.73%	4.89%	0.51%	-0.0928	0.8910	0.7442	0.5136
19	5.94%	3.59%	4.68%	4.76%	4.93%	0.51%	-0.0924	0.8921	0.7464	0.5190
20	5.96%	3.61%	4.70%	4.77%	4.97%	0.51%	-0.0931	0.8930	0.7475	0.5232
21	6.00%	3.63%	4.72%	4.78%	4.99%	0.51%	-0.0908	0.8913	0.7465	0.5262
22	6.02%	3.65%	4.73%	4.79%	5.02%	0.51%	-0.0892	0.8904	0.7458	0.5289
23	6.04%	3.67%	4.74%	4.79%	5.04%	0.51%	-0.0836	0.8906	0.7456	0.5311
24	6.06%	3.68%	4.74%	4.79%	5.06%	0.51%	-0.0749	0.8895	0.7441	0.5303
25	6.06%	3.70%	4.74%	4.78%	5.07%	0.51%	-0.0727	0.8901	0.7447	0.5321
26	6.05%	3.70%	4.74%	4.78%	5.08%	0.50%	-0.0731	0.8897	0.7432	0.5310
27	6.05%	3.70%	4.74%	4.77%	5.08%	0.50%	-0.0728	0.8889	0.7414	0.5295
28	6.04%	3.70%	4.74%	4.77%	5.08%	0.50%	-0.0720	0.8884	0.7394	0.5274
29	6.03%	3.70%	4.73%	4.76%	5.09%	0.50%	-0.0698	0.8878	0.7378	0.5254
30	6.04%	3.69%	4.73%	4.74%	5.10%	0.49%	-0.0662	0.8880	0.7363	0.5221

Table A.1: Descriptive statistics of the spot rates derived from the fixed-for-floating Euro swaps for maturities of 1 to 30 years in the period before September 15th 2008. Autocorrelations for the 30-day (1-month), 90-day (3-months) and 180-day (6-months) lags are represented by $\hat{\rho}$.

Maturity (in years)	Max	Min	Mean	Median	Mode	Standard Deviation	Skewness	$\widehat{\boldsymbol{\rho}}_{\mathbf{30}}$	$\widehat{\boldsymbol{\rho}}_{\mathbf{90}}$	$\widehat{\boldsymbol{\rho}}_{\mathbf{180}}$
1	5.10%	0.17%	1.12%	1.14%	0.41%	0.83%	1.8131	0.7779	0.4891	0.2940
2	4.86%	0.19%	1.27%	1.32%	0.39%	0.83%	1.1022	0.8177	0.5831	0.4170
3	4.82%	0.24%	1.47%	1.51%	0.59%	0.88%	0.7311	0.8472	0.6424	0.4754
4	4.79%	0.31%	1.68%	1.70%	0.98%	0.91%	0.5113	0.8641	0.6736	0.5002
5	4.78%	0.40%	1.88%	1.91%	1.22%	0.91%	0.3534	0.8741	0.6919	0.5149
6	4.77%	0.51%	2.07%	2.07%	1.45%	0.91%	0.2487	0.8805	0.7028	0.5242
7	4.77%	0.63%	2.24%	2.24%	1.66%	0.90%	0.1789	0.8848	0.7096	0.5307
8	4.78%	0.75%	2.39%	2.37%	1.84%	0.88%	0.1389	0.8878	0.7124	0.5338
9	4.80%	0.88%	2.52%	2.48%	2.01%	0.87%	0.1167	0.8894	0.7133	0.5339
10	4.82%	0.99%	2.64%	2.58%	2.15%	0.85%	0.0997	0.8902	0.7129	0.5326
11	4.85%	1.10%	2.74%	2.66%	2.29%	0.84%	0.0887	0.8906	0.7130	0.5311
12	4.89%	1.19%	2.84%	2.74%	3.88%	0.83%	0.0763	0.8909	0.7145	0.5312
13	4.91%	1.28%	2.92%	2.80%	3.91%	0.82%	0.0700	0.8909	0.7146	0.5299
14	4.93%	1.35%	2.98%	2.86%	3.93%	0.82%	0.0670	0.8913	0.7162	0.5294
15	4.95%	1.42%	3.04%	2.90%	3.94%	0.81%	0.0669	0.8917	0.7181	0.5292
16	4.96%	1.47%	3.08%	2.93%	3.94%	0.81%	0.0724	0.8923	0.7200	0.5290
17	4.97%	1.52%	3.10%	2.95%	3.93%	0.80%	0.0780	0.8932	0.7224	0.5305
18	4.97%	1.57%	3.12%	2.95%	3.92%	0.79%	0.0865	0.8940	0.7248	0.5316
19	4.97%	1.60%	3.13%	2.95%	3.90%	0.79%	0.0957	0.8944	0.7266	0.5326
20	4.96%	1.64%	3.13%	2.94%	3.86%	0.78%	0.1071	0.8945	0.7288	0.5330
21	4.95%	1.66%	3.13%	2.92%	3.82%	0.77%	0.1143	0.8938	0.7280	0.5329
22	4.94%	1.68%	3.13%	2.91%	3.76%	0.76%	0.1253	0.8921	0.7258	0.5300
23	4.92%	1.70%	3.11%	2.89%	3.70%	0.74%	0.1372	0.8898	0.7222	0.5253
24	4.91%	1.72%	3.10%	2.88%	3.64%	0.73%	0.1508	0.8870	0.7177	0.5196
25	4.90%	1.73%	3.08%	2.86%	3.58%	0.72%	0.1661	0.8838	0.7130	0.5137
26	4.87%	1.74%	3.07%	2.84%	3.54%	0.70%	0.1822	0.8808	0.7071	0.5059
27	4.86%	1.75%	3.05%	2.83%	3.50%	0.69%	0.1991	0.8773	0.7009	0.4978
28	4.84%	1.75%	3.03%	2.82%	3.46%	0.68%	0.2169	0.8737	0.6943	0.4892
29	4.82%	1.76%	3.01%	2.81%	3.43%	0.66%	0.2333	0.8700	0.6875	0.4804
30	4.81%	1.76%	3.00%	2.79%	3.39%	0.65%	0.2497	0.8656	0.6803	0.4704

Table A.2: Descriptive statistics of the spot rates derived from the fixed-for-floating Euro swaps for maturities of 1 to 30 years in the period after September 15th 2008. Autocorrelations for the 30-day (1-month), 90-day (3-months) and 180-day (6-months) lags are represented by $\hat{\rho}$.

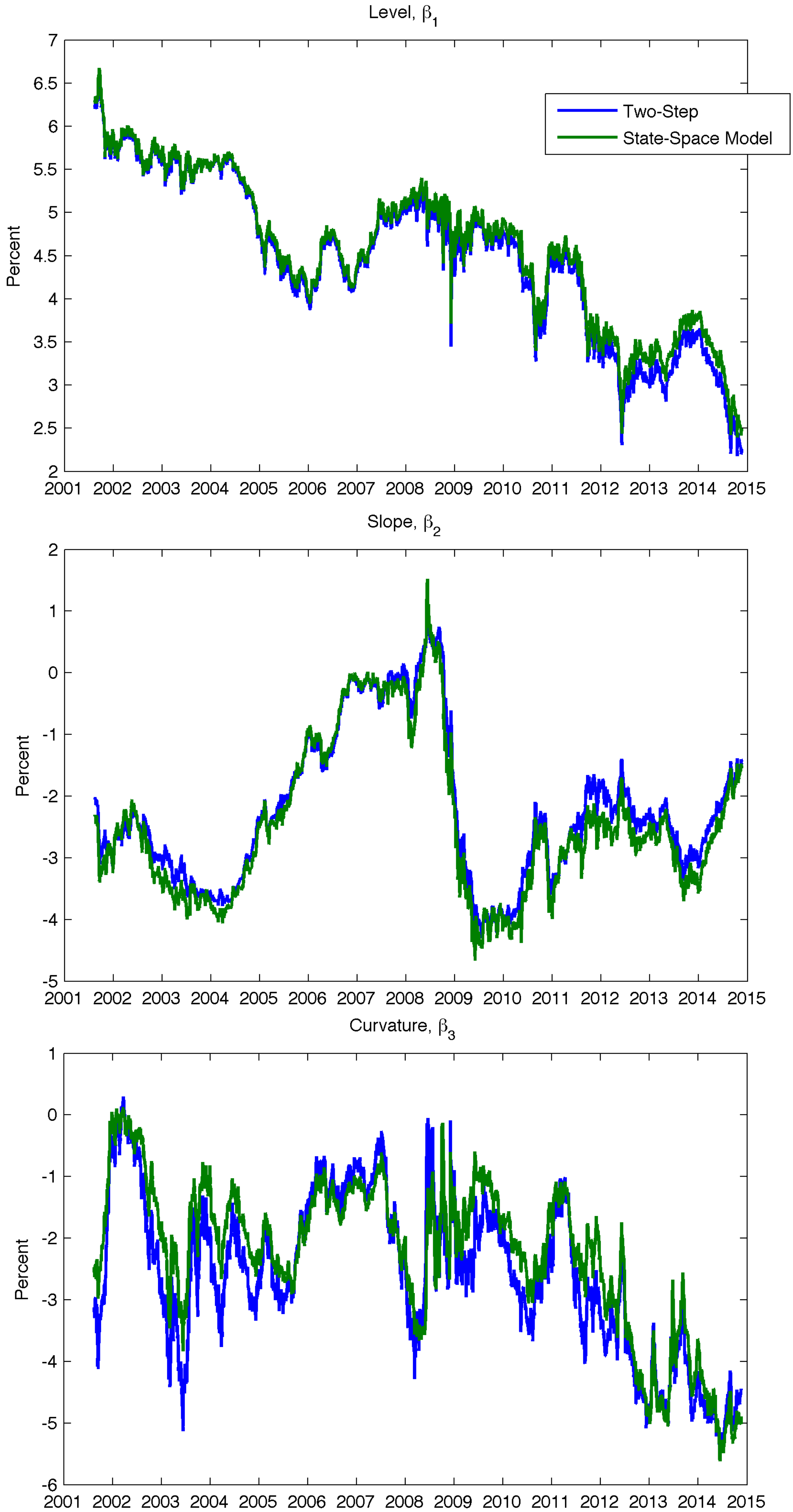


Figure A.3: Comparison of the DNS factor loadings obtained between the one-step and two-step estimations using an LLP of 20 years.

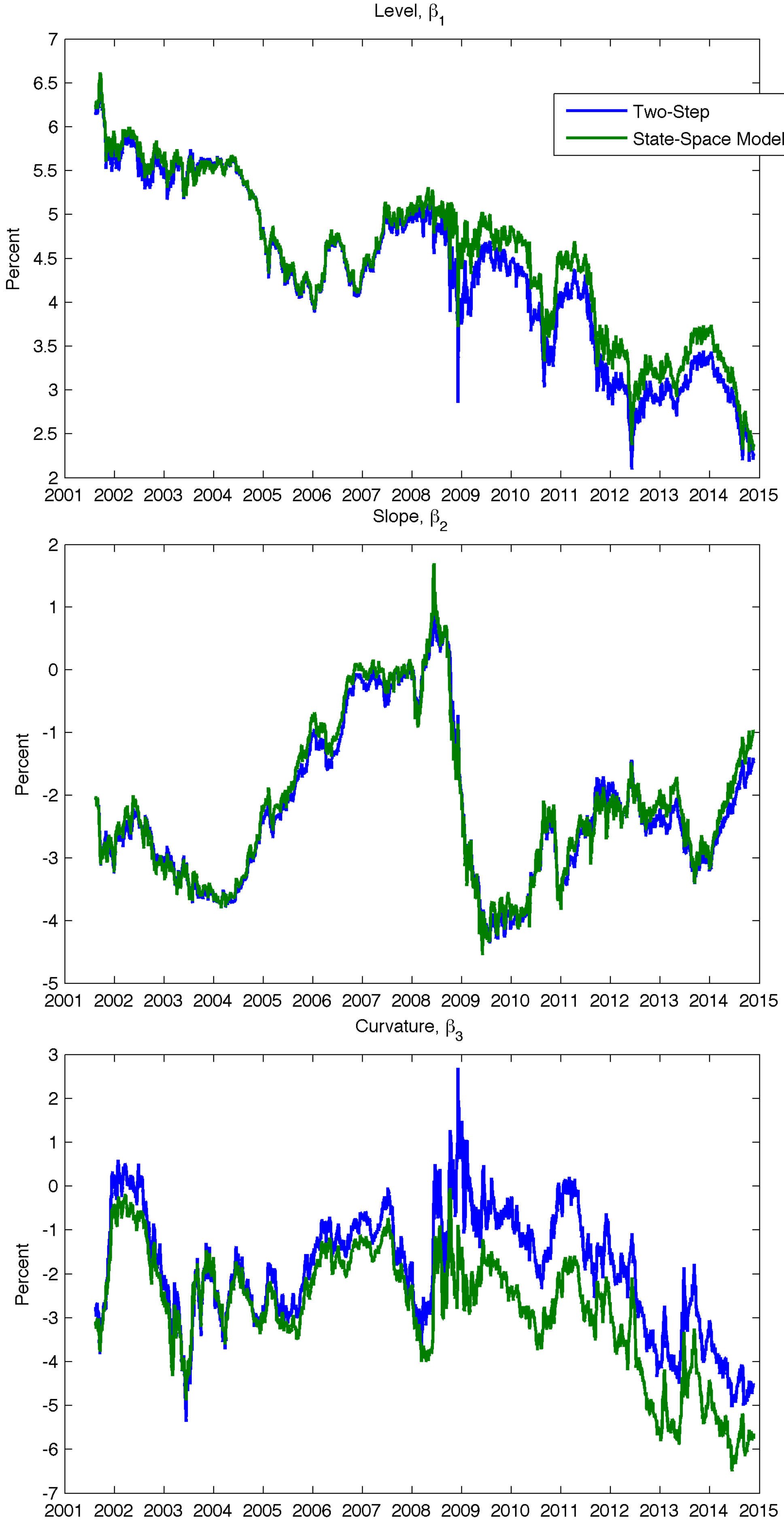
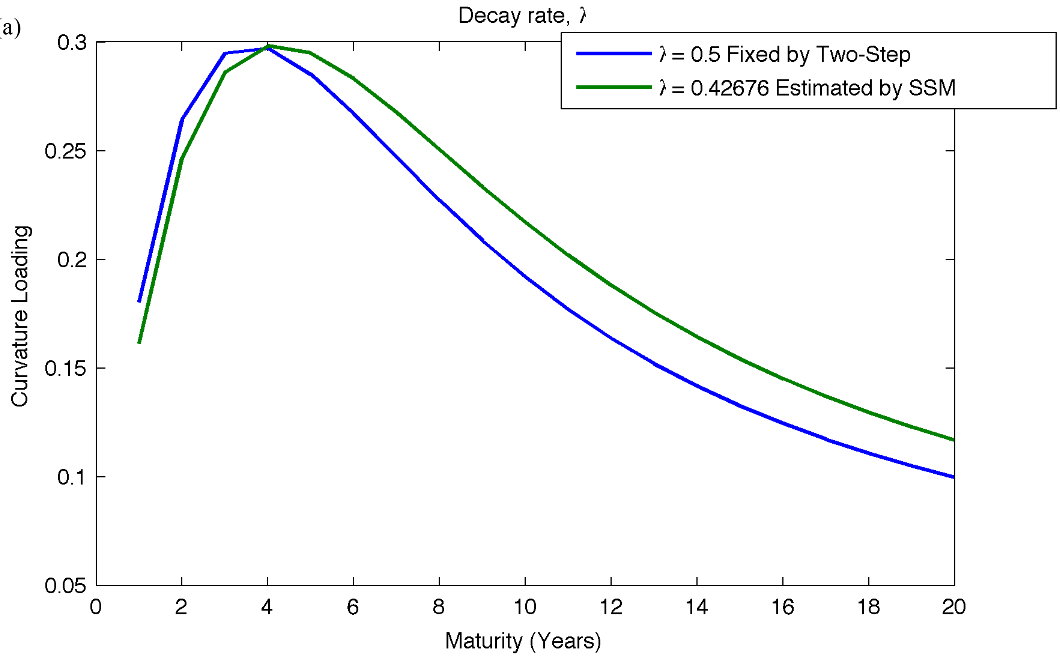


Figure A.4: Comparison of the DNS factor loadings obtained between the one-step and two-step estimations using an LLP of 30 years.

(a)



(b)

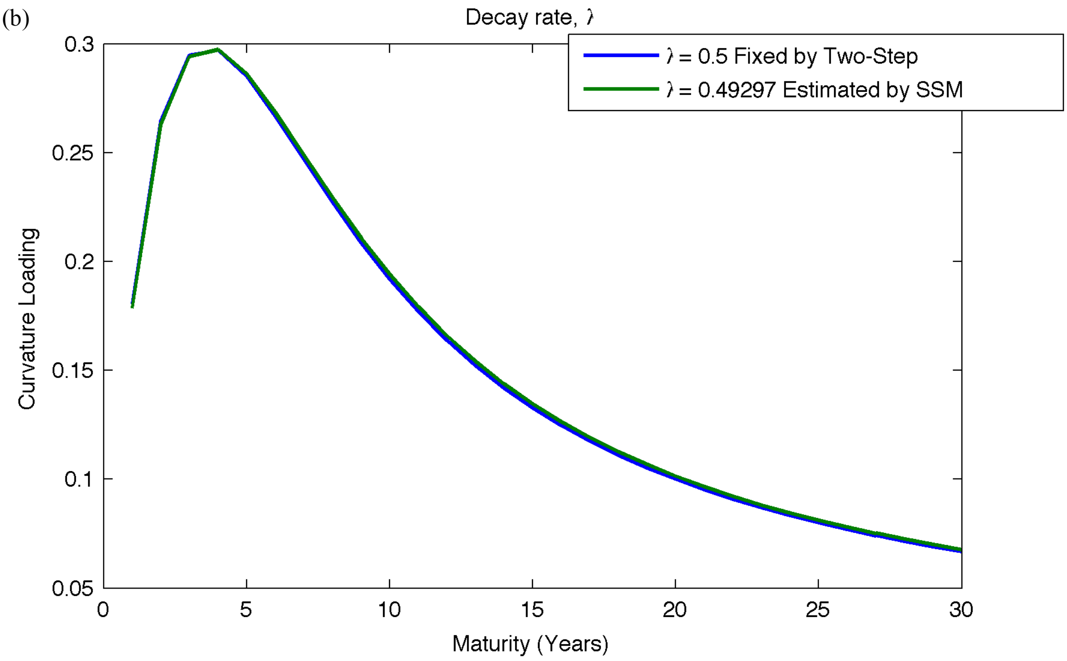


Figure A.5: Comparison of the DNS decay parameters obtained by estimation from the observed data, (a) with an LLP of 20 years, and (b) with an LLP of 30 years, against the fixed decay rate of 0.5 in the one-step DNS.

Maturity (in years)	Mean	Standard Deviation	Min	Max	$\hat{\rho}_{30}$	$\hat{\rho}_{90}$	$\hat{\rho}_{180}$
1	2,1345	1206%	-44,7472	36,6827	0,8133	0,4524	0,1488
2	1,7116	856%	-58,0276	48,3455	0,6916	0,3151	0,1436
3	-0,3602	466%	-30,7731	77,6167	0,3365	0,2219	0,1913
4	0,2881	439%	-71,2483	28,2382	0,1348	0,1170	0,0692
5	-0,0380	476%	-35,6794	86,8689	0,0661	0,0813	0,0320
6	0,1501	561%	-99,2007	58,0375	0,0117	0,0450	0,0213
7	-0,1581	695%	-65,1420	206,1016	0,0079	0,0189	0,0035
8	-0,2298	797%	-242,1144	79,1152	0,0185	0,0081	-0,0204
9	-0,3415	1038%	-96,4660	299,2239	0,0550	0,0194	-0,0280
10	0,2590	1470%	-436,5726	204,1208	0,0524	0,0129	-0,0205
11	0,9713	1784%	-266,1916	393,6883	0,0496	0,0166	0,0005
12	-1,5347	1845%	-206,0150	244,3503	0,0307	0,0118	0,0169
13	1,3366	1735%	-152,2817	277,7716	0,0066	0,0010	0,0285
14	-0,6464	2171%	-223,7584	315,7011	0,0375	0,0048	-0,0002
15	-0,5314	2950%	-433,3515	313,2246	0,0562	0,0138	-0,0206
16	0,5091	2905%	-240,7773	326,4490	0,0617	0,0207	-0,0241
17	-1,0202	2301%	-201,5261	213,5464	0,0346	0,0224	-0,0168
18	2,4032	2066%	-170,4503	147,1512	0,0152	0,0332	0,0059
19	-6,0631	2092%	-202,9970	291,3692	0,1206	0,1189	0,0817
20	4,3121	1022%	-146,2778	104,7202	0,3210	0,2980	0,2455

Table A.6: Descriptive statistics of the z parameters estimated through the Smith-Wilson approach with an LLP of 20 years. As you can see standard deviations are very high especially on the longer maturities and this stems from the fact that this parameters are estimated with the purpose of exactly fitting the observable data.

Maturity (in years)	Mean	Standard Deviation	Min	Max	$\hat{\rho}_{30}$	$\hat{\rho}_{90}$	$\hat{\rho}_{180}$
1	2,1373	1206%	-44,7472	36,6827	0,8134	0,4524	0,1491
2	1,7107	856%	-58,0276	48,3455	0,6919	0,3151	0,1440
3	-0,3602	466%	-30,7731	77,6167	0,3379	0,2214	0,1926
4	0,2874	439%	-71,2483	28,2382	0,1371	0,1156	0,0678
5	-0,0372	476%	-35,6794	86,8689	0,0686	0,0820	0,0254
6	0,1498	561%	-99,2007	58,0375	0,0133	0,0466	0,0158
7	-0,1594	695%	-65,1420	206,1016	0,0109	0,0220	0,0020
8	-0,2277	798%	-242,1144	79,1152	0,0221	0,0114	-0,0203
9	-0,3446	1038%	-96,4664	299,2239	0,0570	0,0206	-0,0278
10	0,2639	1471%	-436,5725	204,1223	0,0531	0,0124	-0,0217
11	0,9680	1786%	-266,1974	393,6880	0,0486	0,0149	-0,0021
12	-1,5371	1849%	-206,0155	244,3728	0,0285	0,0134	0,0168
13	1,3400	1742%	-152,2843	277,7627	0,0063	0,0074	0,0330
14	-0,6416	2177%	-223,7238	315,7353	0,0416	0,0114	0,0000
15	-0,5478	2952%	-433,4843	311,9008	0,0617	0,0193	-0,0251
16	0,5317	2899%	-235,6261	326,9659	0,0665	0,0240	-0,0311
17	-1,0475	2308%	-201,7495	206,4418	0,0374	0,0244	-0,0225
18	2,4644	2444%	-229,5759	177,8360	0,0218	0,0259	0,0017
19	-6,2748	4296%	-494,0791	521,4293	0,0513	0,0270	0,0171
20	9,6699	5699%	-527,7161	690,7111	0,0689	0,0296	0,0346
21	-9,0262	4624%	-517,5161	281,0233	0,0728	0,0219	0,0369
22	5,0569	3741%	-143,9937	1578,9914	0,0183	-0,0181	-0,0251
23	-1,9060	7286%	-3924,2369	384,3964	0,0032	-0,0147	-0,0152
24	-0,5399	11074%	-941,2936	5515,3778	0,0149	-0,0063	0,0026
25	2,1177	10894%	-4315,9177	1312,3027	0,0261	-0,0047	0,0104
26	-2,3868	7272%	-1014,0481	1881,2995	0,0342	-0,0082	0,0051
27	1,0670	4059%	-514,7347	429,7467	0,0341	-0,0110	-0,0208
28	-0,0610	3723%	-387,4729	433,8545	0,0249	-0,0038	-0,0148
29	-4,4848	3872%	-520,7397	352,4197	0,0762	0,0335	0,0184
30	5,1952	1789%	-125,5601	269,4448	0,1913	0,1368	0,1045

Table A.7: Descriptive statistics of the z parameters estimated through the Smith-Wilson approach with an LLP of 30 years. In this case standard deviations become even larger than in the 20-year case.

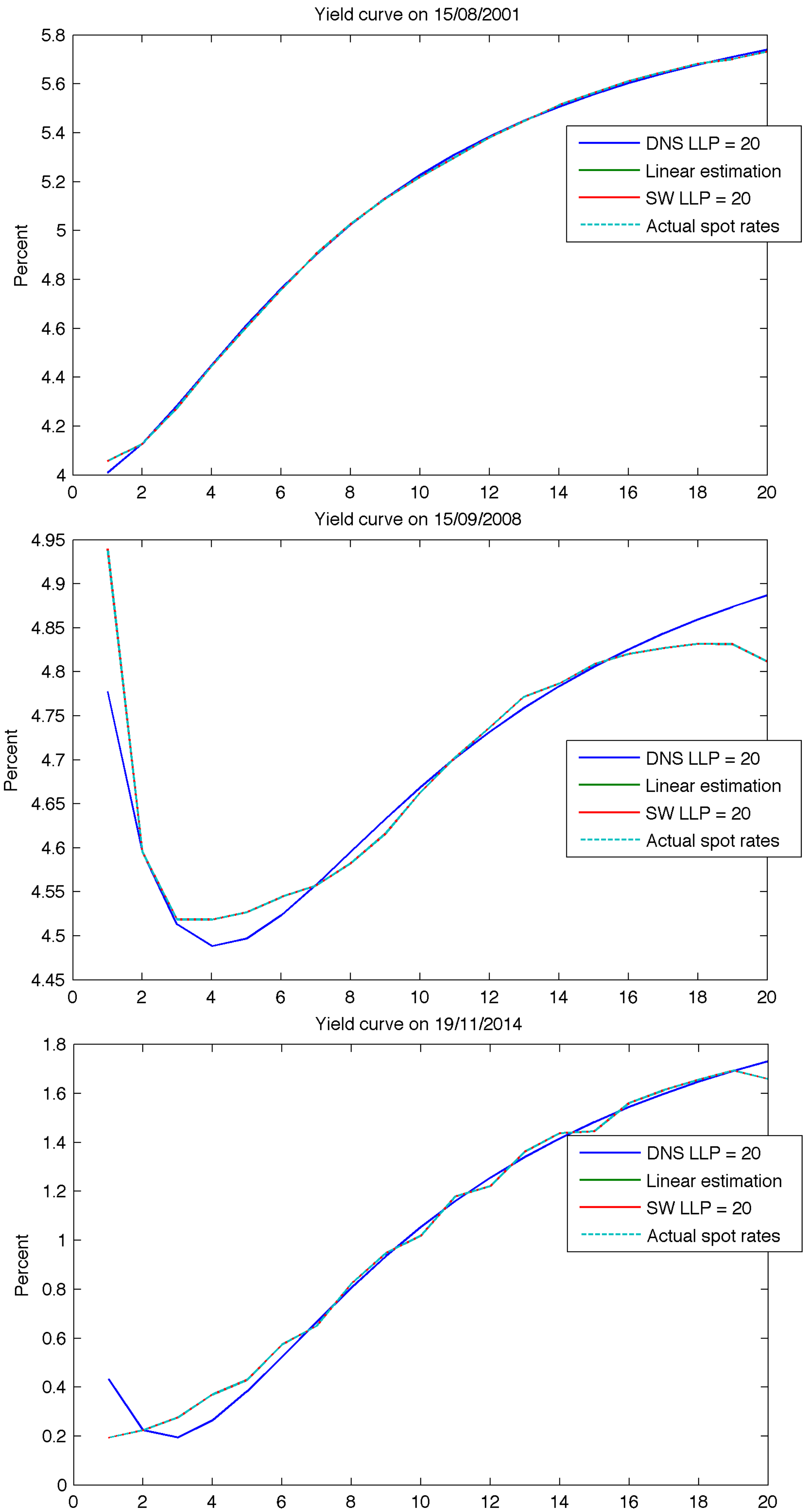


Figure A.8: Comparison of the DNS and SW estimations of the yield curve in-sample with an LLP of 20 years at three different dates: beginning of the sample (August 15th 2001), Lehman bankruptcy (September 15th 2008), and end of the sample (November 19th 2014).

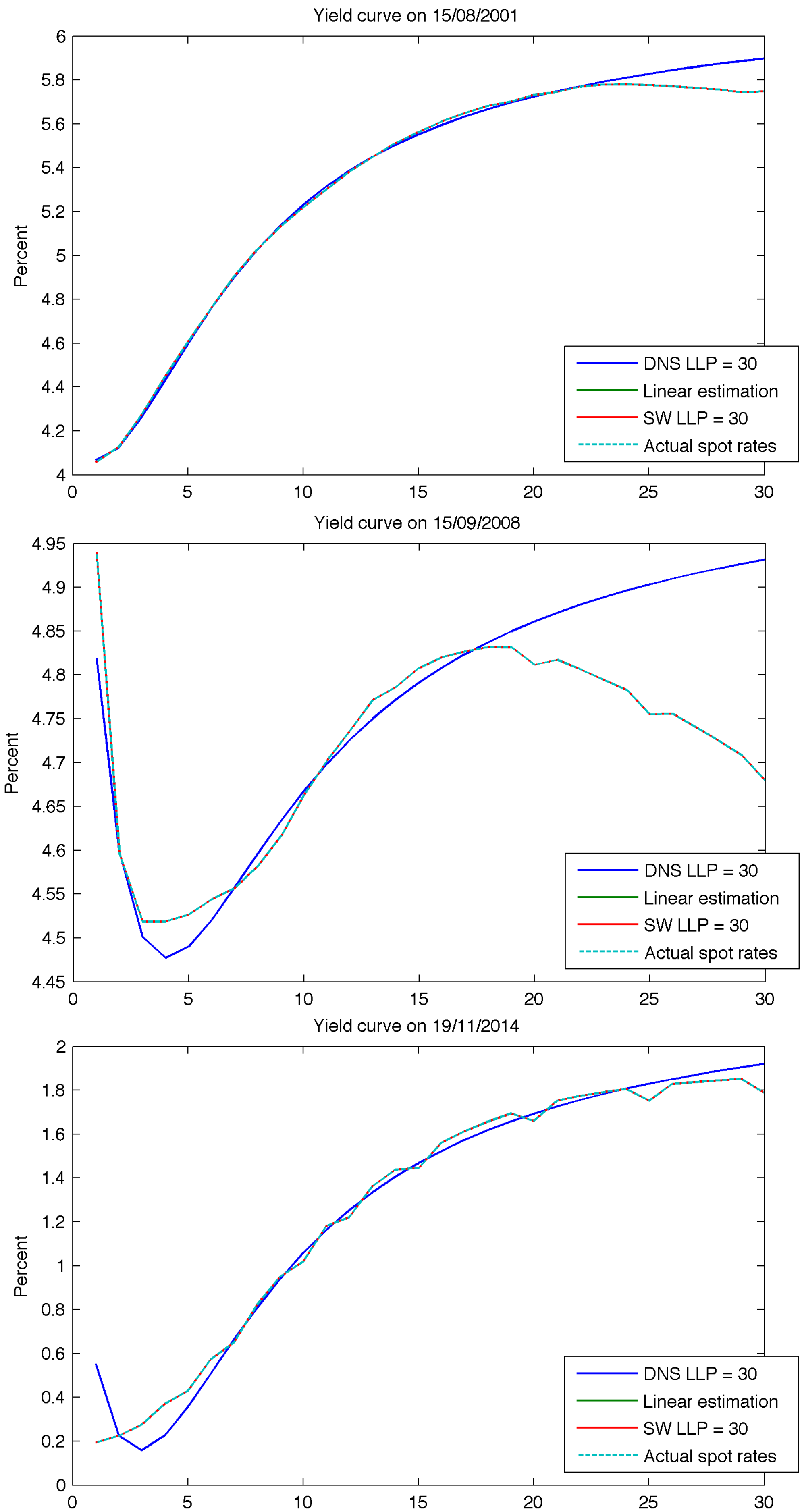


Figure A.9: Comparison of the DNS and SW estimations of the yield curve in-sample with an LLP of 30 years at three different dates: beginning of the sample (August 15th 2001), Lehman bankruptcy (September 15th 2008), and end of the sample (November 19th 2014).

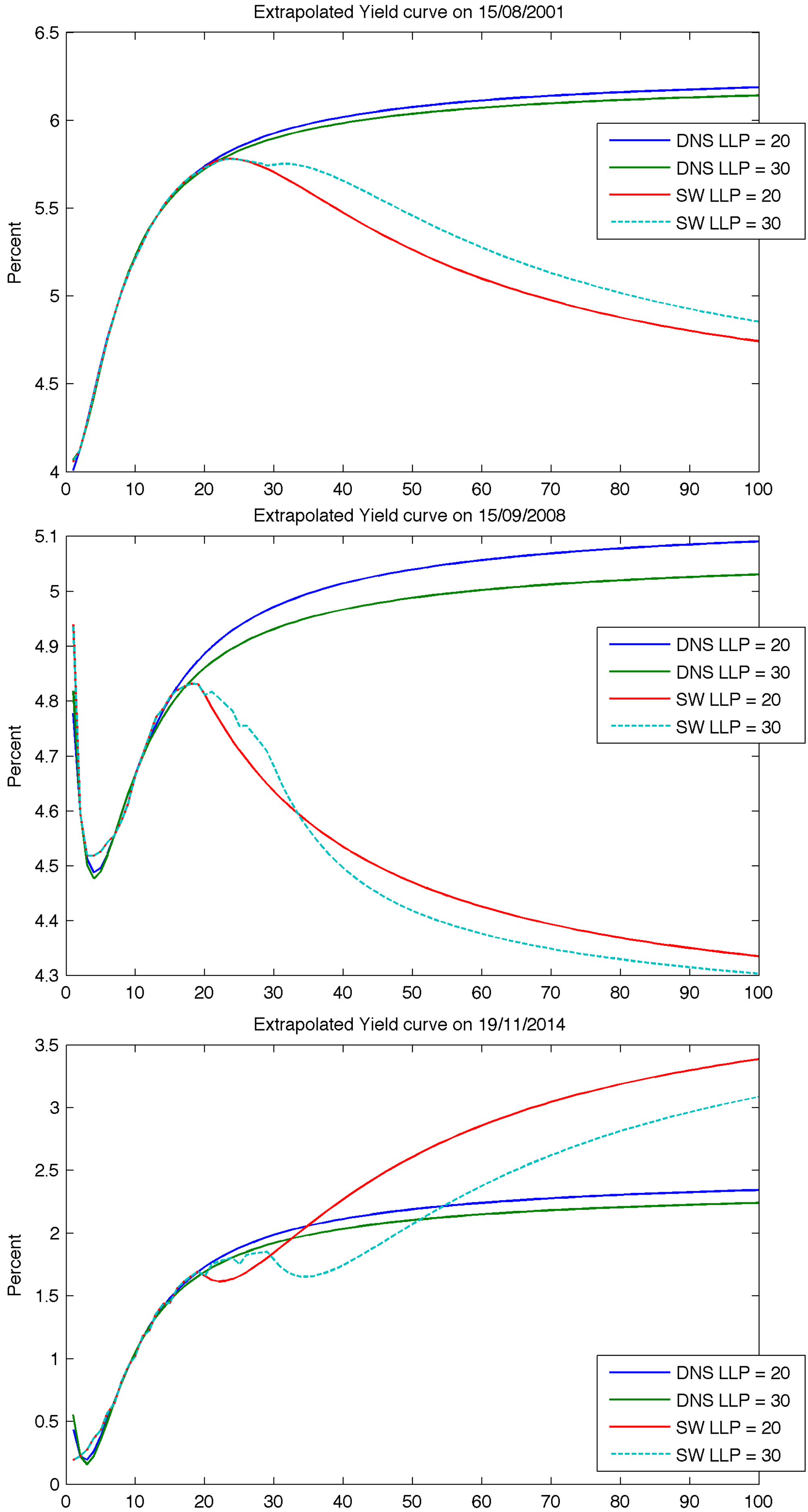


Figure A.10: Extrapolated yield curves by the DNS and SW models up to maturities of 100 years and using both an LLP of 20 and 30 years at three different dates: beginning of the sample (August 15th 2001), Lehman bankruptcy (September 15th 2008), and end of the sample (November 19th 2014).

Maturity (in years)	Mean	Standard Deviation	Min	Max	$\hat{\rho}_{30}$	$\hat{\rho}_{90}$	$\hat{\rho}_{180}$
1	2,25%	1,38%	0,33%	5,75%	0,9687	0,8776	0,7212
2	2,39%	1,33%	0,19%	5,38%	0,9652	0,8730	0,7232
3	2,57%	1,30%	0,17%	5,17%	0,9616	0,8675	0,7202
4	2,74%	1,26%	0,24%	5,06%	0,9587	0,8628	0,7166
5	2,91%	1,22%	0,36%	5,03%	0,9564	0,8591	0,7137
6	3,07%	1,19%	0,49%	5,15%	0,9545	0,8562	0,7117
7	3,20%	1,16%	0,63%	5,24%	0,9529	0,8538	0,7102
8	3,33%	1,13%	0,76%	5,32%	0,9516	0,8518	0,7091
9	3,43%	1,11%	0,89%	5,38%	0,9503	0,8499	0,7083
10	3,53%	1,08%	1,00%	5,43%	0,9491	0,8482	0,7076
11	3,61%	1,06%	1,10%	5,48%	0,9481	0,8466	0,7070
12	3,68%	1,05%	1,20%	5,52%	0,9471	0,8451	0,7064
13	3,75%	1,03%	1,28%	5,56%	0,9461	0,8436	0,7058
14	3,80%	1,02%	1,35%	5,64%	0,9452	0,8422	0,7053
15	3,85%	1,01%	1,42%	5,70%	0,9444	0,8409	0,7047
16	3,90%	1,00%	1,47%	5,76%	0,9436	0,8397	0,7041
17	3,94%	0,99%	1,53%	5,81%	0,9429	0,8385	0,7036
18	3,97%	0,98%	1,57%	5,86%	0,9422	0,8373	0,7030
19	4,00%	0,97%	1,62%	5,90%	0,9415	0,8363	0,7025
20	4,03%	0,97%	1,65%	5,94%	0,9409	0,8353	0,7020
21	4,06%	0,96%	1,69%	5,97%	0,9404	0,8343	0,7015
22	4,08%	0,96%	1,72%	6,01%	0,9398	0,8334	0,7010
23	4,10%	0,95%	1,75%	6,03%	0,9393	0,8326	0,7005
24	4,12%	0,95%	1,78%	6,06%	0,9389	0,8318	0,7001
25	4,14%	0,94%	1,80%	6,09%	0,9384	0,8310	0,6997
26	4,16%	0,94%	1,82%	6,11%	0,9380	0,8303	0,6993
27	4,17%	0,94%	1,85%	6,13%	0,9376	0,8297	0,6989
28	4,19%	0,93%	1,87%	6,15%	0,9373	0,8290	0,6985
29	4,20%	0,93%	1,88%	6,17%	0,9369	0,8284	0,6981
30	4,21%	0,93%	1,90%	6,18%	0,9366	0,8278	0,6978
31	4,23%	0,93%	1,92%	6,20%	0,9363	0,8273	0,6975
32	4,24%	0,92%	1,93%	6,22%	0,9360	0,8268	0,6971
33	4,25%	0,92%	1,95%	6,23%	0,9357	0,8263	0,6968
34	4,26%	0,92%	1,96%	6,24%	0,9354	0,8258	0,6965
35	4,27%	0,92%	1,97%	6,25%	0,9352	0,8253	0,6963
36	4,27%	0,92%	1,98%	6,27%	0,9349	0,8249	0,6960
37	4,28%	0,92%	1,99%	6,28%	0,9347	0,8245	0,6957
38	4,29%	0,91%	2,00%	6,29%	0,9345	0,8241	0,6955
39	4,30%	0,91%	2,01%	6,30%	0,9343	0,8237	0,6952
40	4,31%	0,91%	2,02%	6,31%	0,9341	0,8234	0,6950
41	4,31%	0,91%	2,03%	6,32%	0,9339	0,8230	0,6948
42	4,32%	0,91%	2,04%	6,33%	0,9337	0,8227	0,6946
43	4,32%	0,91%	2,05%	6,33%	0,9335	0,8224	0,6944
44	4,33%	0,91%	2,06%	6,34%	0,9333	0,8221	0,6942
45	4,34%	0,91%	2,07%	6,35%	0,9332	0,8218	0,6940
46	4,34%	0,91%	2,07%	6,36%	0,9330	0,8215	0,6938
47	4,35%	0,90%	2,08%	6,36%	0,9329	0,8212	0,6936
48	4,35%	0,90%	2,09%	6,37%	0,9327	0,8209	0,6934
49	4,36%	0,90%	2,09%	6,38%	0,9326	0,8207	0,6933
50	4,36%	0,90%	2,10%	6,38%	0,9325	0,8204	0,6931
51	4,36%	0,90%	2,10%	6,39%	0,9323	0,8202	0,6929
52	4,37%	0,90%	2,11%	6,39%	0,9322	0,8200	0,6928
53	4,37%	0,90%	2,12%	6,40%	0,9321	0,8198	0,6926

54	4,38%	0,90%	2,12%	6,40%	0,9320	0,8195	0,6925
55	4,38%	0,90%	2,13%	6,41%	0,9318	0,8193	0,6924
56	4,38%	0,90%	2,13%	6,41%	0,9317	0,8191	0,6922
57	4,39%	0,90%	2,13%	6,42%	0,9316	0,8189	0,6921
58	4,39%	0,90%	2,14%	6,42%	0,9315	0,8188	0,6920
59	4,39%	0,90%	2,14%	6,43%	0,9314	0,8186	0,6918
60	4,40%	0,90%	2,15%	6,43%	0,9313	0,8184	0,6917
61	4,40%	0,89%	2,15%	6,43%	0,9312	0,8182	0,6916
62	4,40%	0,89%	2,16%	6,44%	0,9311	0,8181	0,6915
63	4,41%	0,89%	2,16%	6,44%	0,9311	0,8179	0,6914
64	4,41%	0,89%	2,16%	6,45%	0,9310	0,8177	0,6913
65	4,41%	0,89%	2,17%	6,45%	0,9309	0,8176	0,6912
66	4,41%	0,89%	2,17%	6,45%	0,9308	0,8174	0,6911
67	4,42%	0,89%	2,17%	6,46%	0,9307	0,8173	0,6910
68	4,42%	0,89%	2,18%	6,46%	0,9307	0,8171	0,6909
69	4,42%	0,89%	2,18%	6,46%	0,9306	0,8170	0,6908
70	4,42%	0,89%	2,18%	6,47%	0,9305	0,8169	0,6907
71	4,43%	0,89%	2,19%	6,47%	0,9304	0,8167	0,6906
72	4,43%	0,89%	2,19%	6,47%	0,9304	0,8166	0,6905
73	4,43%	0,89%	2,19%	6,47%	0,9303	0,8165	0,6904
74	4,43%	0,89%	2,19%	6,48%	0,9302	0,8164	0,6903
75	4,43%	0,89%	2,20%	6,48%	0,9302	0,8163	0,6903
76	4,44%	0,89%	2,20%	6,48%	0,9301	0,8161	0,6902
77	4,44%	0,89%	2,20%	6,48%	0,9300	0,8160	0,6901
78	4,44%	0,89%	2,20%	6,49%	0,9300	0,8159	0,6900
79	4,44%	0,89%	2,21%	6,49%	0,9299	0,8158	0,6900
80	4,44%	0,89%	2,21%	6,49%	0,9299	0,8157	0,6899
81	4,44%	0,89%	2,21%	6,49%	0,9298	0,8156	0,6898
82	4,45%	0,89%	2,21%	6,50%	0,9298	0,8155	0,6897
83	4,45%	0,89%	2,22%	6,50%	0,9297	0,8154	0,6897
84	4,45%	0,89%	2,22%	6,50%	0,9297	0,8153	0,6896
85	4,45%	0,89%	2,22%	6,50%	0,9296	0,8152	0,6895
86	4,45%	0,89%	2,22%	6,50%	0,9296	0,8151	0,6895
87	4,45%	0,89%	2,22%	6,51%	0,9295	0,8150	0,6894
88	4,46%	0,89%	2,23%	6,51%	0,9295	0,8150	0,6894
89	4,46%	0,89%	2,23%	6,51%	0,9294	0,8149	0,6893
90	4,46%	0,88%	2,23%	6,51%	0,9294	0,8148	0,6892
91	4,46%	0,88%	2,23%	6,51%	0,9293	0,8147	0,6892
92	4,46%	0,88%	2,23%	6,52%	0,9293	0,8146	0,6891
93	4,46%	0,88%	2,24%	6,52%	0,9292	0,8145	0,6891
94	4,46%	0,88%	2,24%	6,52%	0,9292	0,8145	0,6890
95	4,46%	0,88%	2,24%	6,52%	0,9292	0,8144	0,6890
96	4,47%	0,88%	2,24%	6,52%	0,9291	0,8143	0,6889
97	4,47%	0,88%	2,24%	6,52%	0,9291	0,8142	0,6889
98	4,47%	0,88%	2,24%	6,53%	0,9290	0,8142	0,6888
99	4,47%	0,88%	2,25%	6,53%	0,9290	0,8141	0,6888
100	4,47%	0,88%	2,25%	6,53%	0,9290	0,8140	0,6887

Table A.11: Descriptive statistics of the extrapolated yields by the DNS model with a data dependent λ up to maturities of 100 years and using an LLP of 20 years.

Maturity (in years)	Mean	Standard Deviation	Min	Max	$\widehat{\boldsymbol{\rho}}_{30}$	$\widehat{\boldsymbol{\rho}}_{90}$	$\widehat{\boldsymbol{\rho}}_{180}$
1	2,30%	1,36%	0,41%	5,79%	0,9689	0,8783	0,7222
2	2,39%	1,33%	0,19%	5,38%	0,9652	0,8730	0,7232
3	2,55%	1,30%	0,13%	5,16%	0,9614	0,8668	0,7190
4	2,73%	1,27%	0,20%	5,05%	0,9584	0,8620	0,7152
5	2,90%	1,23%	0,33%	5,03%	0,9562	0,8585	0,7126
6	3,06%	1,19%	0,48%	5,15%	0,9544	0,8558	0,7109
7	3,20%	1,16%	0,63%	5,24%	0,9529	0,8537	0,7100
8	3,33%	1,13%	0,77%	5,32%	0,9516	0,8518	0,7093
9	3,44%	1,11%	0,90%	5,38%	0,9504	0,8502	0,7088
10	3,53%	1,08%	1,01%	5,43%	0,9493	0,8486	0,7084
11	3,61%	1,06%	1,11%	5,48%	0,9483	0,8471	0,7080
12	3,68%	1,05%	1,20%	5,51%	0,9473	0,8457	0,7076
13	3,74%	1,04%	1,28%	5,57%	0,9465	0,8444	0,7071
14	3,80%	1,02%	1,35%	5,64%	0,9456	0,8431	0,7067
15	3,84%	1,01%	1,41%	5,70%	0,9449	0,8419	0,7062
16	3,89%	1,00%	1,46%	5,75%	0,9442	0,8408	0,7057
17	3,92%	1,00%	1,51%	5,80%	0,9435	0,8398	0,7053
18	3,95%	0,99%	1,56%	5,85%	0,9429	0,8388	0,7048
19	3,98%	0,98%	1,59%	5,89%	0,9424	0,8379	0,7044
20	4,01%	0,98%	1,63%	5,93%	0,9419	0,8370	0,7039
21	4,04%	0,97%	1,66%	5,96%	0,9414	0,8362	0,7035
22	4,06%	0,97%	1,69%	5,99%	0,9409	0,8354	0,7031
23	4,08%	0,97%	1,72%	6,02%	0,9405	0,8347	0,7027
24	4,10%	0,96%	1,74%	6,04%	0,9401	0,8340	0,7023
25	4,11%	0,96%	1,76%	6,06%	0,9398	0,8334	0,7020
26	4,13%	0,96%	1,78%	6,09%	0,9394	0,8328	0,7016
27	4,14%	0,95%	1,80%	6,10%	0,9391	0,8322	0,7013
28	4,16%	0,95%	1,82%	6,12%	0,9388	0,8317	0,7010
29	4,17%	0,95%	1,84%	6,14%	0,9385	0,8312	0,7007
30	4,18%	0,95%	1,85%	6,16%	0,9382	0,8307	0,7004
31	4,19%	0,95%	1,87%	6,17%	0,9380	0,8303	0,7001
32	4,20%	0,94%	1,88%	6,19%	0,9377	0,8298	0,6999
33	4,21%	0,94%	1,89%	6,20%	0,9375	0,8294	0,6996
34	4,22%	0,94%	1,90%	6,21%	0,9373	0,8290	0,6994
35	4,23%	0,94%	1,91%	6,22%	0,9371	0,8287	0,6991
36	4,24%	0,94%	1,93%	6,23%	0,9369	0,8283	0,6989
37	4,24%	0,94%	1,94%	6,24%	0,9367	0,8280	0,6987
38	4,25%	0,93%	1,94%	6,25%	0,9365	0,8276	0,6985
39	4,26%	0,93%	1,95%	6,26%	0,9363	0,8273	0,6983
40	4,26%	0,93%	1,96%	6,27%	0,9362	0,8270	0,6981
41	4,27%	0,93%	1,97%	6,28%	0,9360	0,8268	0,6979
42	4,28%	0,93%	1,98%	6,29%	0,9359	0,8265	0,6978
43	4,28%	0,93%	1,99%	6,30%	0,9357	0,8262	0,6976
44	4,29%	0,93%	1,99%	6,30%	0,9356	0,8260	0,6974
45	4,29%	0,93%	2,00%	6,31%	0,9355	0,8257	0,6973
46	4,30%	0,93%	2,01%	6,32%	0,9353	0,8255	0,6971
47	4,30%	0,93%	2,01%	6,32%	0,9352	0,8253	0,6970
48	4,31%	0,93%	2,02%	6,33%	0,9351	0,8251	0,6968
49	4,31%	0,92%	2,02%	6,34%	0,9350	0,8249	0,6967
50	4,32%	0,92%	2,03%	6,34%	0,9349	0,8247	0,6966
51	4,32%	0,92%	2,03%	6,35%	0,9348	0,8245	0,6964

52	4,32%	0,92%	2,04%	6,35%	0,9347	0,8243	0,6963
53	4,33%	0,92%	2,04%	6,36%	0,9346	0,8241	0,6962
54	4,33%	0,92%	2,05%	6,36%	0,9345	0,8239	0,6961
55	4,33%	0,92%	2,05%	6,37%	0,9344	0,8238	0,6960
56	4,34%	0,92%	2,06%	6,37%	0,9343	0,8236	0,6959
57	4,34%	0,92%	2,06%	6,37%	0,9342	0,8235	0,6957
58	4,34%	0,92%	2,07%	6,38%	0,9341	0,8233	0,6956
59	4,35%	0,92%	2,07%	6,38%	0,9340	0,8232	0,6955
60	4,35%	0,92%	2,07%	6,39%	0,9340	0,8230	0,6955
61	4,35%	0,92%	2,08%	6,39%	0,9339	0,8229	0,6954
62	4,35%	0,92%	2,08%	6,39%	0,9338	0,8227	0,6953
63	4,36%	0,92%	2,08%	6,40%	0,9337	0,8226	0,6952
64	4,36%	0,92%	2,09%	6,40%	0,9337	0,8225	0,6951
65	4,36%	0,92%	2,09%	6,40%	0,9336	0,8224	0,6950
66	4,36%	0,92%	2,09%	6,41%	0,9335	0,8222	0,6949
67	4,37%	0,92%	2,10%	6,41%	0,9335	0,8221	0,6948
68	4,37%	0,92%	2,10%	6,41%	0,9334	0,8220	0,6948
69	4,37%	0,92%	2,10%	6,42%	0,9334	0,8219	0,6947
70	4,37%	0,91%	2,11%	6,42%	0,9333	0,8218	0,6946
71	4,38%	0,91%	2,11%	6,42%	0,9332	0,8217	0,6945
72	4,38%	0,91%	2,11%	6,43%	0,9332	0,8216	0,6945
73	4,38%	0,91%	2,11%	6,43%	0,9331	0,8215	0,6944
74	4,38%	0,91%	2,12%	6,43%	0,9331	0,8214	0,6943
75	4,38%	0,91%	2,12%	6,43%	0,9330	0,8213	0,6943
76	4,38%	0,91%	2,12%	6,44%	0,9330	0,8212	0,6942
77	4,39%	0,91%	2,12%	6,44%	0,9329	0,8211	0,6941
78	4,39%	0,91%	2,12%	6,44%	0,9329	0,8210	0,6941
79	4,39%	0,91%	2,13%	6,44%	0,9328	0,8209	0,6940
80	4,39%	0,91%	2,13%	6,44%	0,9328	0,8209	0,6940
81	4,39%	0,91%	2,13%	6,45%	0,9328	0,8208	0,6939
82	4,39%	0,91%	2,13%	6,45%	0,9327	0,8207	0,6939
83	4,40%	0,91%	2,14%	6,45%	0,9327	0,8206	0,6938
84	4,40%	0,91%	2,14%	6,45%	0,9326	0,8205	0,6937
85	4,40%	0,91%	2,14%	6,45%	0,9326	0,8205	0,6937
86	4,40%	0,91%	2,14%	6,46%	0,9326	0,8204	0,6936
87	4,40%	0,91%	2,14%	6,46%	0,9325	0,8203	0,6936
88	4,40%	0,91%	2,14%	6,46%	0,9325	0,8203	0,6935
89	4,40%	0,91%	2,15%	6,46%	0,9324	0,8202	0,6935
90	4,41%	0,91%	2,15%	6,46%	0,9324	0,8201	0,6934
91	4,41%	0,91%	2,15%	6,47%	0,9324	0,8201	0,6934
92	4,41%	0,91%	2,15%	6,47%	0,9323	0,8200	0,6934
93	4,41%	0,91%	2,15%	6,47%	0,9323	0,8199	0,6933
94	4,41%	0,91%	2,15%	6,47%	0,9323	0,8199	0,6933
95	4,41%	0,91%	2,16%	6,47%	0,9322	0,8198	0,6932
96	4,41%	0,91%	2,16%	6,47%	0,9322	0,8197	0,6932
97	4,41%	0,91%	2,16%	6,47%	0,9322	0,8197	0,6931
98	4,41%	0,91%	2,16%	6,48%	0,9321	0,8196	0,6931
99	4,42%	0,91%	2,16%	6,48%	0,9321	0,8196	0,6931
100	4,42%	0,91%	2,16%	6,48%	0,9321	0,8195	0,6930

Table A.12: Descriptive statistics of the extrapolated yields by the DNS model with a data dependent λ up to maturities of 100 years and using an LLP of 30 years.

Maturity (in years)	Mean	Standard Deviation	Min	Max	$\widehat{\boldsymbol{\rho}}_{30}$	$\widehat{\boldsymbol{\rho}}_{90}$	$\widehat{\boldsymbol{\rho}}_{180}$
1	2,24%	1,37%	0,17%	5,33%	0,9707	0,8837	0,7252
2	2,39%	1,33%	0,19%	5,38%	0,9652	0,8730	0,7233
3	2,58%	1,31%	0,24%	5,26%	0,9620	0,8683	0,7230
4	2,75%	1,27%	0,31%	5,13%	0,9594	0,8643	0,7199
5	2,92%	1,23%	0,40%	5,06%	0,9569	0,8599	0,7153
6	3,07%	1,19%	0,51%	5,14%	0,9547	0,8563	0,7119
7	3,20%	1,16%	0,63%	5,24%	0,9529	0,8536	0,7099
8	3,32%	1,13%	0,75%	5,31%	0,9515	0,8520	0,7095
9	3,43%	1,10%	0,88%	5,38%	0,9504	0,8506	0,7095
10	3,52%	1,08%	0,99%	5,42%	0,9493	0,8488	0,7087
11	3,61%	1,06%	1,10%	5,48%	0,9481	0,8470	0,7075
12	3,68%	1,04%	1,19%	5,51%	0,9469	0,8451	0,7062
13	3,75%	1,03%	1,28%	5,56%	0,9459	0,8431	0,7049
14	3,81%	1,02%	1,35%	5,65%	0,9452	0,8419	0,7047
15	3,85%	1,01%	1,42%	5,71%	0,9448	0,8413	0,7052
16	3,89%	1,01%	1,47%	5,81%	0,9443	0,8412	0,7065
17	3,92%	1,01%	1,52%	5,87%	0,9444	0,8418	0,7087
18	3,94%	1,01%	1,57%	5,91%	0,9446	0,8426	0,7112
19	3,96%	1,01%	1,60%	5,94%	0,9451	0,8439	0,7140
20	3,97%	1,02%	1,64%	5,96%	0,9458	0,8454	0,7171
21	3,98%	1,02%	1,63%	5,99%	0,9462	0,8466	0,7195
22	3,99%	1,01%	1,62%	6,02%	0,9463	0,8474	0,7214
23	4,00%	1,01%	1,62%	6,05%	0,9463	0,8480	0,7228
24	4,00%	0,99%	1,63%	6,06%	0,9462	0,8484	0,7239
25	4,01%	0,98%	1,66%	6,07%	0,9461	0,8486	0,7247
26	4,02%	0,97%	1,69%	6,07%	0,9459	0,8487	0,7254
27	4,03%	0,95%	1,72%	6,07%	0,9457	0,8488	0,7259
28	4,03%	0,94%	1,76%	6,12%	0,9455	0,8488	0,7263
29	4,04%	0,92%	1,80%	6,19%	0,9453	0,8488	0,7267
30	4,04%	0,91%	1,85%	6,25%	0,9451	0,8488	0,7269
31	4,05%	0,89%	1,89%	6,29%	0,9449	0,8488	0,7272
32	4,05%	0,87%	1,93%	6,33%	0,9447	0,8487	0,7273
33	4,06%	0,85%	1,98%	6,36%	0,9445	0,8486	0,7275
34	4,06%	0,84%	2,02%	6,38%	0,9443	0,8486	0,7276
35	4,07%	0,82%	2,07%	6,39%	0,9441	0,8485	0,7277
36	4,07%	0,80%	2,11%	6,40%	0,9440	0,8484	0,7278
37	4,08%	0,79%	2,15%	6,40%	0,9438	0,8483	0,7278
38	4,08%	0,77%	2,19%	6,40%	0,9437	0,8483	0,7279
39	4,08%	0,76%	2,23%	6,39%	0,9435	0,8482	0,7279
40	4,09%	0,74%	2,27%	6,38%	0,9434	0,8482	0,7279
41	4,09%	0,73%	2,31%	6,36%	0,9433	0,8481	0,7280
42	4,09%	0,71%	2,35%	6,35%	0,9432	0,8480	0,7280
43	4,09%	0,70%	2,38%	6,33%	0,9431	0,8480	0,7280
44	4,10%	0,69%	2,42%	6,31%	0,9430	0,8479	0,7280
45	4,10%	0,67%	2,45%	6,29%	0,9430	0,8479	0,7280
46	4,10%	0,66%	2,49%	6,27%	0,9429	0,8478	0,7280
47	4,10%	0,65%	2,52%	6,24%	0,9428	0,8478	0,7280
48	4,11%	0,64%	2,55%	6,22%	0,9428	0,8478	0,7281
49	4,11%	0,62%	2,58%	6,19%	0,9427	0,8477	0,7281
50	4,11%	0,61%	2,61%	6,17%	0,9427	0,8477	0,7281
51	4,11%	0,60%	2,64%	6,14%	0,9426	0,8477	0,7281
52	4,11%	0,59%	2,66%	6,11%	0,9426	0,8477	0,7281
53	4,11%	0,58%	2,69%	6,09%	0,9425	0,8476	0,7281

54	4,12%	0,57%	2,72%	6,06%	0,9425	0,8476	0,7281
55	4,12%	0,56%	2,74%	6,04%	0,9425	0,8476	0,7281
56	4,12%	0,55%	2,77%	6,01%	0,9425	0,8476	0,7281
57	4,12%	0,54%	2,79%	5,99%	0,9424	0,8476	0,7281
58	4,12%	0,53%	2,81%	5,96%	0,9424	0,8476	0,7281
59	4,12%	0,53%	2,84%	5,94%	0,9424	0,8475	0,7281
60	4,12%	0,52%	2,86%	5,91%	0,9424	0,8475	0,7281
61	4,13%	0,51%	2,88%	5,89%	0,9424	0,8475	0,7281
62	4,13%	0,50%	2,90%	5,86%	0,9423	0,8475	0,7281
63	4,13%	0,49%	2,92%	5,84%	0,9423	0,8475	0,7281
64	4,13%	0,49%	2,94%	5,82%	0,9423	0,8475	0,7281
65	4,13%	0,48%	2,96%	5,80%	0,9423	0,8475	0,7281
66	4,13%	0,47%	2,98%	5,77%	0,9423	0,8475	0,7281
67	4,13%	0,46%	2,99%	5,75%	0,9423	0,8475	0,7281
68	4,13%	0,46%	3,01%	5,73%	0,9423	0,8475	0,7281
69	4,13%	0,45%	3,03%	5,71%	0,9423	0,8475	0,7281
70	4,14%	0,45%	3,04%	5,69%	0,9423	0,8475	0,7281
71	4,14%	0,44%	3,06%	5,67%	0,9423	0,8475	0,7281
72	4,14%	0,43%	3,08%	5,65%	0,9423	0,8475	0,7281
73	4,14%	0,43%	3,09%	5,63%	0,9423	0,8475	0,7281
74	4,14%	0,42%	3,11%	5,61%	0,9423	0,8475	0,7281
75	4,14%	0,42%	3,12%	5,59%	0,9422	0,8475	0,7281
76	4,14%	0,41%	3,13%	5,58%	0,9422	0,8474	0,7281
77	4,14%	0,41%	3,15%	5,56%	0,9422	0,8474	0,7281
78	4,14%	0,40%	3,16%	5,54%	0,9422	0,8474	0,7281
79	4,14%	0,39%	3,17%	5,53%	0,9422	0,8474	0,7281
80	4,14%	0,39%	3,19%	5,51%	0,9422	0,8474	0,7281
81	4,14%	0,39%	3,20%	5,49%	0,9422	0,8474	0,7281
82	4,14%	0,38%	3,21%	5,48%	0,9422	0,8474	0,7281
83	4,15%	0,38%	3,22%	5,46%	0,9422	0,8474	0,7281
84	4,15%	0,37%	3,23%	5,45%	0,9422	0,8474	0,7281
85	4,15%	0,37%	3,25%	5,43%	0,9422	0,8474	0,7281
86	4,15%	0,36%	3,26%	5,42%	0,9422	0,8474	0,7281
87	4,15%	0,36%	3,27%	5,41%	0,9422	0,8474	0,7281
88	4,15%	0,35%	3,28%	5,39%	0,9422	0,8474	0,7281
89	4,15%	0,35%	3,29%	5,38%	0,9422	0,8474	0,7281
90	4,15%	0,35%	3,30%	5,37%	0,9422	0,8474	0,7281
91	4,15%	0,34%	3,31%	5,35%	0,9422	0,8474	0,7281
92	4,15%	0,34%	3,32%	5,34%	0,9422	0,8474	0,7281
93	4,15%	0,34%	3,33%	5,33%	0,9422	0,8474	0,7281
94	4,15%	0,33%	3,34%	5,32%	0,9422	0,8474	0,7281
95	4,15%	0,33%	3,35%	5,31%	0,9422	0,8474	0,7281
96	4,15%	0,33%	3,35%	5,29%	0,9422	0,8474	0,7281
97	4,15%	0,32%	3,36%	5,28%	0,9422	0,8474	0,7281
98	4,15%	0,32%	3,37%	5,27%	0,9422	0,8474	0,7281
99	4,15%	0,32%	3,38%	5,26%	0,9422	0,8474	0,7281
100	4,15%	0,31%	3,39%	5,25%	0,9422	0,8474	0,7281

Table A.13: Descriptive statistics of the extrapolated yields by the SW model up to maturities of 100 years and using an LLP of 20 years.

Maturity (in years)	Mean	Standard Deviation	Min	Max	$\hat{\rho}_{30}$	$\hat{\rho}_{90}$	$\hat{\rho}_{180}$
1	2,24%	1,37%	0,17%	5,33%	0,9707	0,8837	0,7252
2	2,39%	1,33%	0,19%	5,38%	0,9652	0,8730	0,7232
3	2,58%	1,31%	0,24%	5,26%	0,9620	0,8683	0,7229
4	2,75%	1,27%	0,31%	5,13%	0,9594	0,8642	0,7198
5	2,92%	1,23%	0,40%	5,06%	0,9569	0,8598	0,7153
6	3,07%	1,19%	0,51%	5,14%	0,9547	0,8562	0,7118
7	3,20%	1,16%	0,63%	5,24%	0,9528	0,8536	0,7099
8	3,32%	1,13%	0,75%	5,31%	0,9515	0,8519	0,7094
9	3,43%	1,10%	0,88%	5,38%	0,9504	0,8505	0,7095
10	3,52%	1,08%	0,99%	5,42%	0,9493	0,8488	0,7086
11	3,61%	1,06%	1,10%	5,48%	0,9480	0,8469	0,7075
12	3,68%	1,04%	1,19%	5,51%	0,9469	0,8450	0,7061
13	3,75%	1,03%	1,28%	5,56%	0,9459	0,8430	0,7048
14	3,81%	1,02%	1,35%	5,65%	0,9452	0,8419	0,7047
15	3,85%	1,01%	1,42%	5,71%	0,9448	0,8413	0,7051
16	3,89%	1,01%	1,47%	5,81%	0,9443	0,8412	0,7065
17	3,92%	1,01%	1,52%	5,87%	0,9444	0,8417	0,7087
18	3,94%	1,01%	1,57%	5,91%	0,9446	0,8426	0,7112
19	3,96%	1,01%	1,60%	5,94%	0,9451	0,8439	0,7139
20	3,97%	1,02%	1,64%	5,96%	0,9458	0,8454	0,7170
21	3,98%	1,02%	1,66%	6,00%	0,9457	0,8457	0,7188
22	3,98%	1,02%	1,68%	6,02%	0,9459	0,8464	0,7210
23	3,98%	1,02%	1,70%	6,04%	0,9463	0,8471	0,7227
24	3,98%	1,03%	1,72%	6,06%	0,9464	0,8475	0,7240
25	3,97%	1,03%	1,73%	6,06%	0,9468	0,8484	0,7259
26	3,96%	1,03%	1,74%	6,05%	0,9471	0,8492	0,7274
27	3,95%	1,03%	1,75%	6,05%	0,9473	0,8498	0,7288
28	3,94%	1,04%	1,75%	6,04%	0,9475	0,8504	0,7301
29	3,93%	1,04%	1,76%	6,03%	0,9477	0,8511	0,7314
30	3,92%	1,04%	1,76%	6,04%	0,9479	0,8516	0,7322
31	3,91%	1,03%	1,73%	6,06%	0,9480	0,8519	0,7327
32	3,91%	1,03%	1,69%	6,08%	0,9479	0,8520	0,7331
33	3,90%	1,02%	1,66%	6,10%	0,9477	0,8520	0,7333
34	3,90%	1,01%	1,65%	6,10%	0,9475	0,8520	0,7334
35	3,90%	1,00%	1,65%	6,11%	0,9472	0,8519	0,7335
36	3,90%	0,99%	1,66%	6,13%	0,9469	0,8518	0,7335
37	3,90%	0,98%	1,68%	6,14%	0,9466	0,8516	0,7335
38	3,90%	0,97%	1,70%	6,14%	0,9464	0,8515	0,7335
39	3,91%	0,95%	1,72%	6,14%	0,9461	0,8513	0,7334
40	3,91%	0,94%	1,75%	6,14%	0,9458	0,8512	0,7334
41	3,91%	0,92%	1,78%	6,13%	0,9456	0,8510	0,7333
42	3,91%	0,91%	1,81%	6,12%	0,9454	0,8509	0,7332
43	3,92%	0,90%	1,84%	6,11%	0,9452	0,8507	0,7331
44	3,92%	0,88%	1,87%	6,10%	0,9450	0,8506	0,7331
45	3,93%	0,87%	1,90%	6,08%	0,9448	0,8505	0,7330
46	3,93%	0,85%	1,94%	6,06%	0,9446	0,8503	0,7329
47	3,93%	0,84%	1,97%	6,04%	0,9444	0,8502	0,7329
48	3,94%	0,83%	2,01%	6,02%	0,9443	0,8501	0,7328
49	3,94%	0,81%	2,04%	6,00%	0,9442	0,8500	0,7327
50	3,94%	0,80%	2,07%	5,98%	0,9440	0,8499	0,7327
51	3,95%	0,79%	2,11%	5,96%	0,9439	0,8499	0,7326
52	3,95%	0,77%	2,14%	5,94%	0,9438	0,8498	0,7326
53	3,96%	0,76%	2,17%	5,92%	0,9437	0,8497	0,7326

54	3,96%	0,75%	2,20%	5,89%	0,9436	0,8497	0,7325
55	3,96%	0,74%	2,23%	5,87%	0,9436	0,8496	0,7325
56	3,97%	0,72%	2,26%	5,85%	0,9435	0,8496	0,7325
57	3,97%	0,71%	2,29%	5,83%	0,9434	0,8495	0,7324
58	3,97%	0,70%	2,32%	5,80%	0,9434	0,8495	0,7324
59	3,98%	0,69%	2,35%	5,78%	0,9433	0,8494	0,7324
60	3,98%	0,68%	2,38%	5,76%	0,9433	0,8494	0,7323
61	3,98%	0,67%	2,40%	5,74%	0,9432	0,8494	0,7323
62	3,99%	0,66%	2,43%	5,72%	0,9432	0,8493	0,7323
63	3,99%	0,65%	2,46%	5,70%	0,9432	0,8493	0,7323
64	3,99%	0,64%	2,48%	5,68%	0,9431	0,8493	0,7323
65	4,00%	0,63%	2,51%	5,66%	0,9431	0,8493	0,7323
66	4,00%	0,62%	2,53%	5,64%	0,9431	0,8492	0,7323
67	4,00%	0,61%	2,55%	5,62%	0,9431	0,8492	0,7322
68	4,00%	0,60%	2,58%	5,60%	0,9430	0,8492	0,7322
69	4,01%	0,60%	2,60%	5,58%	0,9430	0,8492	0,7322
70	4,01%	0,59%	2,62%	5,56%	0,9430	0,8492	0,7322
71	4,01%	0,58%	2,64%	5,54%	0,9430	0,8492	0,7322
72	4,01%	0,57%	2,66%	5,53%	0,9430	0,8492	0,7322
73	4,02%	0,56%	2,68%	5,51%	0,9430	0,8491	0,7322
74	4,02%	0,56%	2,70%	5,49%	0,9429	0,8491	0,7322
75	4,02%	0,55%	2,72%	5,47%	0,9429	0,8491	0,7322
76	4,02%	0,54%	2,74%	5,46%	0,9429	0,8491	0,7322
77	4,03%	0,54%	2,76%	5,44%	0,9429	0,8491	0,7322
78	4,03%	0,53%	2,78%	5,43%	0,9429	0,8491	0,7322
79	4,03%	0,52%	2,80%	5,41%	0,9429	0,8491	0,7322
80	4,03%	0,52%	2,81%	5,40%	0,9429	0,8491	0,7322
81	4,03%	0,51%	2,83%	5,38%	0,9429	0,8491	0,7322
82	4,04%	0,50%	2,85%	5,37%	0,9429	0,8491	0,7322
83	4,04%	0,50%	2,86%	5,36%	0,9429	0,8491	0,7322
84	4,04%	0,49%	2,88%	5,34%	0,9429	0,8491	0,7322
85	4,04%	0,49%	2,89%	5,33%	0,9429	0,8491	0,7322
86	4,04%	0,48%	2,91%	5,32%	0,9429	0,8491	0,7322
87	4,05%	0,47%	2,92%	5,30%	0,9429	0,8491	0,7322
88	4,05%	0,47%	2,94%	5,29%	0,9429	0,8491	0,7322
89	4,05%	0,46%	2,95%	5,28%	0,9429	0,8491	0,7322
90	4,05%	0,46%	2,96%	5,27%	0,9429	0,8491	0,7322
91	4,05%	0,45%	2,98%	5,26%	0,9429	0,8491	0,7322
92	4,05%	0,45%	2,99%	5,24%	0,9429	0,8491	0,7322
93	4,06%	0,44%	3,00%	5,23%	0,9429	0,8491	0,7322
94	4,06%	0,44%	3,02%	5,22%	0,9429	0,8491	0,7322
95	4,06%	0,43%	3,03%	5,21%	0,9429	0,8491	0,7322
96	4,06%	0,43%	3,04%	5,20%	0,9429	0,8491	0,7322
97	4,06%	0,43%	3,05%	5,19%	0,9429	0,8491	0,7322
98	4,06%	0,42%	3,06%	5,18%	0,9429	0,8491	0,7322
99	4,06%	0,42%	3,08%	5,17%	0,9429	0,8491	0,7322
100	4,07%	0,41%	3,09%	5,16%	0,9429	0,8491	0,7322

Table A.14: Descriptive statistics of the extrapolated yields by the SW model up to maturities of 100 years and using an LLP of 30 years.